Network Contraction for Rapid Equilibrium Assessment

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1 Introduction

Static user equilibrium traffic assignment lies at the core of a variety of transportation planning and operations models. When a large number of alternatives are considered, a frequent practice is to use a *sketch network* containing a specific region of interest, while assuming unchanged conditions elsewhere. This allows a very large number of alternatives to be compared in short order, but taking the *ceteris paribus* assumption loses important global diversion/attraction effects which occur due to local changes, and makes it difficult to evaluate the large-scale impacts of the alternatives.

Network contraction provides an alternate approach for rapid equilibrium sensitivity analysis. The idea is to replace a large transportation network with a smaller one which responds similarly to changes in input flow. A simple example is compressing two links which are in series into a single link by simply adding their cost functions. It is often useful to keep a portion of the network uncontracted (to "zoom in" on an area of particular interest) while contracting the remainder of the network to reduce computation time.

These transformations, derived formally in the next section, closely resemble techniques used in the analysis of electric circuits and, indeed, the formulas bear much similarity although derived from first principles here. Two important distinctions between these two domains are that (1) the difference in electric potential between two nodes must be the same among *all* paths, whereas the travel cost between two nodes must be equal only among *used* paths; and (2) in transportation networks, travelers departing different origins are distinct and may see different "potentials." This distinction vanishes if we consider a single origin at a time, and restrict attention to the set of minimum-cost paths rooted at that origin. More precisely, given an origin u, we only consider the subgraph $B^u = (N, A^u)$ where $A^u = \{(i,j) \in A : t_{ij} - (\pi_j - \pi_i) = 0$ and $\boldsymbol{\pi}$ is a set of shortest-path node potentials from u. Following Dial (2006), we term this an *equilibrium bush*.

Within an origin's equilibrium bush, all paths between two nodes have equal travel cost, and we seek to analyze how the equilibrium travel cost will change between each origin and destination either as a function of the input flows, or as a function of changes in link parameters in uncontracted regions of the graph. We assume that (1) the equilibrium bush remains unchanged throughout the range of input demands, and (2) the network is planar.

2 Network Transformations

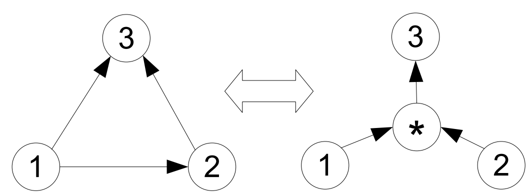

Figure 1: Schematic of the delta-Y and Y-delta transforms.

The following notation is used: let $t_{ij}(x_{ij})$ represent the travel time on link (i,j) when the demand for travel on this link is x_{ij}, and let $T_{ij}(X_{ij})$ represent the equilibrium travel cost between any two nodes among *all* paths connecting these nodes, as a function of the total demand for travel between these nodes (regardless of destination). In particular, we are interested in the derivatives $T'_{ij}(X_{ij})$; if we have a "base" equilibrium solution, we know X_{ij} and T_{ij}, and can make the approximation $T_{ij}(X) \approx T_{ij}(X_{ij}) + T'_{ij}(X_{ij})(X - X_{ij})$. Higher-order approximations naturally follow if further derivatives are known. Our goal, therefore, is to derive T'_{ij} for a variety of network configurations. This necessarily involves calculating the derivatives in equilibrium link flows $\alpha_{k\ell} \equiv dx_{k\ell}/dX_{ij}$ as demand between i and j varies.

Series Consider two links (i,j) and (j,k) in series, with cost functions $t_{ij}(x_{ij})$ and $t_{jk}(x_{jk})$, and let x be the demand for travel between i and k. Then $x_{ij} = x_{jk} = x$, and the equilibrium travel time T_{ik} between i and k is clearly $(t_{ij} + t_{jk})(X_{ik})$, so $T'_{ik} = (t'_{ij} + t'_{jk})(X_{ik})$. From this point on, dependence of link travel times on flows is suppressed for brevity.

Parallel Consider two links $(i,j)^1$ and $(i,j)^2$ in parallel, with cost functions t^1_{ij} and t^2_{ij}Because the bush is in equilibrium we have $t^1_{ij} = t^2_{ij}$. Furthermore, as X_{ij} changes, x^1_{ij} and x^2_{ij} will change such that the equilibrium is preserved, that is,

$$\frac{dT_{ij}}{dX_{ij}} = \frac{dt^1_{ij}}{dX_{ij}} = \frac{dt^2_{ij}}{dX_{ij}} \iff \frac{dt^1_{ij}}{dx^1_{ij}}\alpha^1_{ij} = \frac{dt^2_{ij}}{dx^1_{ij}}\alpha^2_{ij} \tag{1}$$

Furthermore, by conservation of flow we have $\alpha^1_{ij} + \alpha^2_{ij} = 1$, so by substitution $\alpha^1_{ij} = (t^2_{ij})'/[(t^1_{ij})' + (t^2_{ij})']$ and

$$\frac{dT_{ij}}{dx} = \frac{(t^1_{ij})'(t^2_{ij})'}{(t^1_{ij})' + (t^2_{ij})'} \tag{2}$$

Delta-Y Consider an undirected cycle of three nodes with an empty interior (Figure 1). Without loss of generality let these be nodes 1, 2, and 3, and let their node potentials satisfy $\pi_1 \leq \pi_2 \leq \pi_3$. The arc orientations must then be as indicated as in the figure. For brevity, let Δ_1, Δ_2, and Δ_3 represent the marginal inflow to the cycle; that is, $\alpha_{12} + \alpha_{13} = \Delta_1$, $-\alpha_{12} + \alpha_{23} = \Delta_2$, and $-\alpha_{13} - \alpha_{32} = \Delta_3$. These equations are linearly dependent, as flow conservation demands. However, the requirement that the bush remain at equilibrium also requires $\alpha_{13}t'_{13} = \alpha_{12}t'_{12} + \alpha_{23}t'_{23}$. This provides a third, linearly independent equation to solve for α_{12}, α_{13}, and α_{23}. Omitting the details, we have

$$\alpha_{12} = \frac{t'_{13}\Delta_1 - t'_{23}\Delta_2}{t'_{12} + t'_{13} + t'_{23}} \qquad \alpha_{13} = \frac{t'_{12}\Delta_1 - t'_{23}\Delta_3}{t'_{12} + t'_{13} + t'_{23}} \qquad \alpha_{23} = \frac{t'_{12}\Delta_2 - t'_{13}\Delta_3}{t'_{12} + t'_{13} + t'_{23}}$$

with a pleasing symmetry. Thus, the differential change in equilibrium costs are given by

$$dT_{12} = t'_{12}\frac{t'_{13}\Delta_1 - t'_{23}\Delta_2}{t'_{12} + t'_{13} + t'_{23}} \qquad dT_{13} = t'_{13}\frac{t'_{12}\Delta_1 - t'_{23}\Delta_3}{t'_{12} + t'_{13} + t'_{23}} \qquad dT_{23} = t'_{23}\frac{t'_{12}\Delta_2 - t'_{13}\Delta_3}{t'_{12} + t'_{13} + t'_{23}}$$

Now, consider the Y junction in Figure 1. The geometry of this junction forces $\alpha_{1*} = \Delta_1$, $\alpha_{2*} = \Delta_2$, and $\alpha_{*3} = -\Delta_3$. If we choose delay functions t_{1*}, t_{2*}, and t_{*3} such that

$$t'_{1*} = \frac{t'_{12}t'_{13}}{t'_{12} + t'_{13} + t'_{23}} \qquad t'_{2*} = \frac{t'_{12}t'_{23}}{t'_{12} + t'_{13} + t'_{23}} \qquad t'_{*3} = \frac{t'_{13}t'_{23}}{t'_{12} + t'_{13} + t'_{23}}$$

it is easily seen that the change in equilibrium costs is identical to that in the delta.

Y-Delta The delta-Y equations can be inverted to reverse the previous transformation, allowing one to replace a three-pronged intersection with an equivalent triangular component. The reader can verify that the following equations indeed accomplish this inversion.

$$t'_{12} = \frac{t'_{1*}t'_{2*} + t'_{1*}t'_{*3} + t'_{2*}t'_{*3}}{t'_{*3}} \qquad t'_{13} = \frac{t'_{1*}t'_{2*} + t'_{1*}t'_{*3} + t'_{2*}t'_{*3}}{t'_{*2}} \qquad t'_{23} = \frac{t'_{1*}t'_{2*} + t'_{1*}t'_{*3} + t'_{2*}t'_{*3}}{t'_{*1}}$$

3 Demonstration

This section demonstrates the above procedure on the well-known Braess network as portrayed in Sheffi (1985). Figure 2 shows iteratively how the contraction is performed: panel (a) shows the equilibrium solution on the initial network with a travel demand of 6 between nodes 1 and 4, along with link travel times and travel time derivatives. Panel (b) shows the link travel time derivatives alone; panel (c) shows the derivatives after applying a delta-Y transform to nodes 2, 3, and 4; panel (d) shows the result of series transformations to nodes 1, 2, and * and 1, 3, and *; panel (e) shows the result of a parallel transformation among nodes 1 and *; and panel (f) shows the result after a final series contraction (note that Feo and Provan (1993) show that reduction to a single link is always possible, as long as there is just one origin and one destination), demonstrating that the derivative of the equilibrium travel time between nodes 1 and 4, with respect to travel demand, is 31/13. To verify this, Figure 3 plots the equilibrium travel time for a variety of travel demand

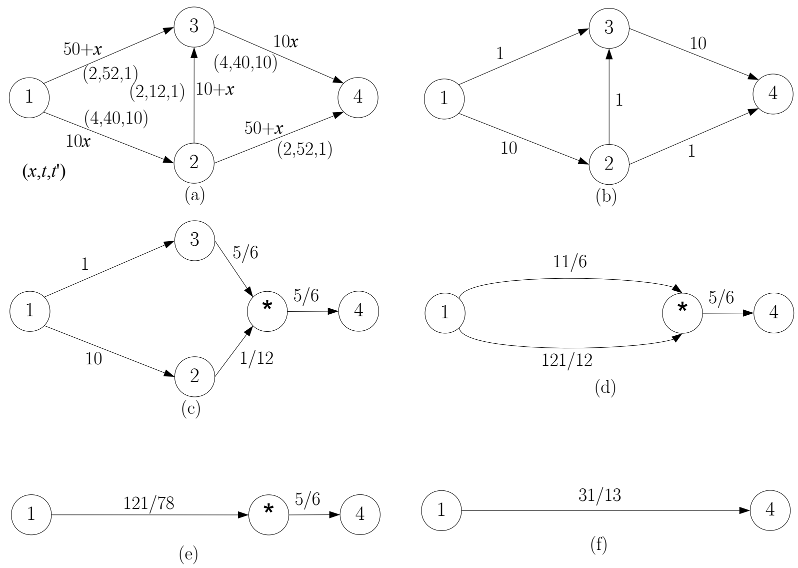

Figure 2: Application to the Braess network.

values; the resulting graph is piecewise linear due to the cost functions used in this network. When travel demand lies in the interval between 40/11 and 100/9, the approximation is exact. More comprehensive demonstrations, including multiple origins and destinations, is included in the full paper.

References

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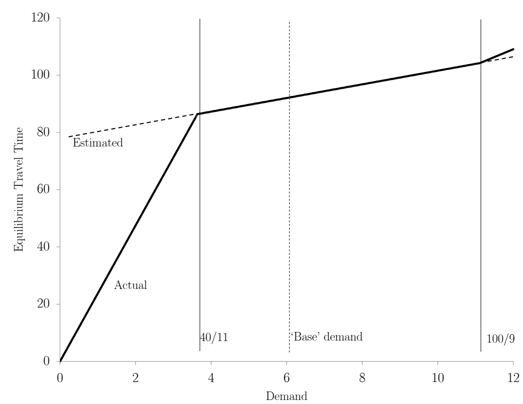

Figure 3: Application to the Braess network.