Fundamentals of Probability

CE 311S

WHY PROBABILITY?

In the first half of the class, we will assume that we know how a random process works, and then investigate what we would expect to see in the world.

P(H) = 0.5
P(T) = 0.5

HTTHHTTTTHHTHTHH

This is **probability**.

Most real-world problems go in the opposite direction (starting from evidence, and then trying to reconstruct how the random process is working).

HTTHHTTTTTTTHTHT $\Rightarrow$ P(H) = ?
P(T) = ?

This is **statistics**. But we can't do this properly until we understand the first process (probability).

Eventually, we want to answer questions like:

- Is the coin fair?
- How many coin flips do I need to determine whether the coin is fair?
- Are the odds of the coin changing over time?
- How long will I have to wait before I see three heads in a row?
- ... and so on.

...but first we need to set the "ground rules."

ELEMENTARY SET THEORY

Basic probability concepts can be cast in set-theoretic terms

Let A and B be two events from the universal set S ($A \subset S$ and $B \subset S$)

The notation $x \in A$ means that the outcome x is part of the event A (x is an element of A)

For a concrete example, let S contain all possible outcomes from three coin flips, let A be the set of outcomes where "the first flip is heads", and B "exactly two heads were thrown." What are $\mathcal{S}$, A, and B?

The **union** of two sets A and B is written $A \cup B$.

If $C = A \cup B$, then this means that every element of C is an element of A or an element of B (possibly both)

With our specific examples, what is $A \cup B$?

Think of unions as **OR**

The **intersection** of two sets A and B is written $A \cap B$.

If $C = A \cap B$, then this means that every element of C is an element of A and an element of B

With our specific examples, what is $A \cap B$?

Think of intersections as **AND**

The **complement** of a set A is written A^c

A^c consists of all the elements of the universal set which are *not* elements of A.

With our specific exmaples, what is A^c?

Think of complements as **NOT**

The **difference** of sets A and B is written $A - B$

$A - B$ contains all of the elements which are in A, but not in B.

With our specific exmaples, what is $A - B$?

If $A \cap B = \emptyset$, then A and B are *mutually exclusive* or *disjoint*. (These terms are synonyms.)

Example: The sets of outcomes "the first flip is heads" and "the first flip is tails" are mutually exclusive.

A collection of sets is **exhaustive** if their union is the sample space S.

Example: The sets of outcomes "at least one flip is heads" and "at least one flip is tails" are exhaustive.

A collection of events is a **partition** of S if they are mutually exclusive and exhaustive.

Example: The sets of outcomes “the first flip is heads” and “the first flip is tails” are exhaustive.

We can combine these operations:

Example: What does $(A \cup B)^c$ mean?

It is very easy to get confused when these operations are combined. A visual technique called Venn diagrams can help.

Example: What are A, B, $A \cup B$, $A \cap B$, and A^c in Venn diagrams?

Don't feel like you need to draw everything all at once. You can apply Venn diagrams in steps:

Example: What is $(A \cup B)^c$ What is $(A \cap B)^c \cup (B \cap C)$?

If there are three sets A, B, and C, how can we express the set whose elements belong to *at most two* of the three? (Section 1.3.6, problem 1d)

PROBABILITY FUNDAMENTALS

Consider an "experiment" — any activity or process whose outcome is unknown or uncertain.

- Flipping a coin
- Flipping a coin twice
- The five-card poker hand you'll be dealt
- The rainfall in Austin next month
- The grades you'll get in courses this semester

An **outcome** is a possible result of an experiment.

- Flipping a coin: heads (H) or tails (T)
- Two flips: two H's or T's (HH, TH, etc.)
- Poker: five specific cards
- Rainfall: a single real number
- Grades: one grade (A, A-, etc.) for each class you're in

One of our first tasks will be to list and count all outcomes.

The **sample space** S is the set of all possible outcomes.

- Flipping a coin: $S = \{H, T\}$
- Two flips: $S = \{HH, TH, HT, TT\}$
- Poker: the set of all five-card hands
- Rainfall: $S = \mathbb{R}^+$
- Grades: ?

An **event** is a subset of the sample space.

A set E is a subset of S if every element of E is also an element of S

Examples (*not* unique, there are multiple events for every sample space):

- Two flips: "one head and one tails" : $\{TH, HT\}$
- Poker: the set of all full houses
- Rainfall: the interval $[20, 30]$
- Grades: the set of grades with A's or B's in every course
- Flipping a coin: ?

The empty set $\emptyset$ and S are always events; these are "special cases." $\emptyset$ is called the **null event** and $\mathcal{S}$ is sometimes called the **universe**.

Sometimes events have an intuitive description, but they don't have to:

- The set of all full houses
- The set $\{(5\heartsuit, 3\heartsuit, 2\clubsuit, K\diamondsuit, K\heartsuit), (3\clubsuit, 2\clubsuit, A\clubsuit, 10\spadesuit, 3\spadesuit)\}$

An event "occurs" if the outcome is an element of that event.

Review: what is the difference between outcomes, events, and the sample space?

AXIOMS OF PROBABILITY

Axioms are fundamental truths from which everything else can be proven. (Axioms are taken for granted and do not need to be proved themselves.)

Axioms are used in formal mathematics and logic; I'll provide intuitive descriptions as well, but if there is any doubt or ambiguity here, the formal mathematics take priority.

The **probability** of an event A is meant to precisely describe the chance that A will occur. This value is denoted $P(A)$, and will range between 0 and 1. What are the ground rules for dealing with probability?

The three axioms of probability are as follows.

Let $\mathcal{S}$ be the sample space.

1. For any event, $P(A) \geq 0$. (A negative probability makes no sense.)
2. $P(\mathcal{S}) = 1$. (Certainly one of the elements in the sample space will be observed.)
3. If $A = \{A_i\}$ is a collection of disjoint events, then $P(\cup A_i) = \sum_i P(A_i)$.

These can all be interpreted geometrically in terms of Venn diagrams.

Examples of the third axiom:

1. If A_1 and A_2 are disjoint, $P(A_1 \cup A_2) = P(A_1) + P(A_2)$
2. A_1 is rolling a 3 on a die; A_2 is rolling a 4.
3. A_1 is rolling a 1 or 2; A_2 is rolling a multiple of 3.

Why don't we have an axiom that $P(A) \leq 1$

We want the set of axioms to be as *small as possible*. We can prove $P(A) \leq 1$ from the other three axioms, by contradiction:

For any event A, A and A^c are disjoint, so by Axiom 3 $P(A \cup A^c) = P(A) + P(A^c)$. Also, $A \cup A^c = \mathcal{S}$, so by Axiom 2 $P(A) + P(A^c) = 1$. Now if $P(A) > 1$, then $P(A^c) < 0$, which violates Axiom 1. Therefore $P(A) > 1$ is impossible, so $P(A) \leq 1$.

A slightly trickier proof

Prove that $P(\emptyset) = 0$, where $\emptyset$ is the "null event" (the event containing no outcomes at all.)

The text gives an alternative proof of the second fact. I've given a different one for variety, and to show another approach. Any (logically valid) proof will do.

What are the valid probabilities?

Consider a (possibly weighted) coin which can land either heads (H) or tails (T). What are all of the events? Are the following probabilities valid?

1. $P(H) = P(T) = 1/2$
2. $P(H) = 1/3,\ P(T) = 2/3$
3. $P(H) = 1/2,\ P(T) = 2/3$
4. $P(H) = 0,\ P(T) = 1$
5. $P(H) = -1,\ P(T) = 2$

Axioms do not completely determine probabilities; they just lay the ground rules for what's valid and what's not.

Remember that $P(A) = 1 - P(A^c)$.

This is useful, because even if we are interested in $P(A)$, sometimes it is easier to calculate the probability that A *doesn't happen* (that is, A^c). No problem, just subtract that from 1.

Inclusion-exclusion principle for 2 events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Think about this in terms of Venn diagrams.

Example 1.11

There is a 60% chance it will rain today; a 50% chance it will rain tomorrow; and a 30% chance it will not rain on either day.

1. What is the probability it will rain today or tomorrow?
2. What is the probability it will rain today and tomorrow?
3. What is the probability it will rain today but not tomorrow?
4. What is the probability it will rain today or tomorrow, but not both?

What is $P(A \cup B \cup C)$?

Can you find the general formula when there are many events? (Can you even draw a Venn diagram with 4+ events?)