Elementary Combinatorics

CE 311S

INTRODUCTION

How can we actually calculate probabilities?

Let's assume that there all of the outcomes in the sample space S are equally likely. If $|A|$ is the number of outcomes included in the event A, then

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In this case, calculating probabilities is reduced to *counting*.

For situations with a small number of outcomes, we can count directly.

Example: A family has three children; what is the probability that exactly two of them are boys?

For situations with a larger number of outcomes, we need something more systematic.

Example: In five-card draw poker, what's the probability of being dealt a royal flush?

MULTIPLICATION RULE FOR COUNTING

The Multiplication Rule for Counting

If we have a set of n_1 objects, and a set of n_2 objects, the number of ways to choose one object from each set is $n_1 \times n_2$.

The same rule applies where there are more than two sets.

LUNCH MENU

Pick a sandwich from column A and a soup from column B

Column A	Column B
Ham and Cheese	Minestrone
Pastrami	Chicken Noodle
Hummus Pita	Chili
Tuna	

How many different lunches can I have?

LUNCH MENU

Pick a sandwich from column A and a soup from column B

Column A	**Column B**
Ham and Cheese	Minestrone
Pastrami	Chicken Noodle
Hummus Pita	Chili
Tuna	

There are 4 different sandwiches, each of which can be paired with 3 different soups, so there are $4 \times 3 = 12$ possible lunches.

What if there are more than two options?

SOMEWHAT SIMPLER LUNCH MENU

Pick a salad from column A, a sandwich from column A and a soup from column C

Column A	**Column B**	**Column C**
Caesar	Ham and Cheese	Minestrone
House	Pastrami	Chicken Noodle
	Hummus Pita	Chili
	Tuna	

The multiplication rule still applies: $2 \times 3 \times 4 = 24$ possible lunches.

If we are choosing k things from the *same set*, and we can choose the same thing multiple times, then the multiplication rule gives us n^k possibilities (assuming order matters).

In how many ways can we assign birthdays to students in this class?

PERMUTATIONS (ORDERED WITHOUT REPLACEMENT)

Let's shift gears here... we'll now talk about choosing multiple items from a set X without any duplications. Examples:

- Dealing a hand from a deck of cards
- Choosing a starting lineup for a sports team
- Borrowing books from the library

This is frequently called sampling *without replacement*. Unlike the previous examples, we are picking multiple items from the *same* set. However, the fundamental principle still applies.

Given a set A, a **permutation** is an *ordered* subset of A.

Example: To discourage cheating, a professor develops 10 exam questions. Each exam will consist of four of these questions in a different order. How many different exams can be created?

The following logic shows how the multiplication rule can be applied

- The first question can be any of the 10 questions developed
- The second question can be any of the 9 questions which are not the first
- The third question can be any of the remaining 8
- The fourth question can be any of the remaining 7

The following logic shows how the multiplication rule can be applied

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- The fourth question can be any of the remaining 7

Therefore, there are $10 \times 9 \times 8 \times 7 = 5040$ possible exams.

In general, for a set X containing n elements, the number of permutations of size k is given by:

$$P_k^n = (n)(n-1)\cdots(n-k+1)$$

This can also be written as

$$P_k^n = \prod_{i=0}^{k-1}(n-i)$$

or

$$P_k^n = \frac{n!}{(n-k)!}$$

A factorial sidebar

$n!$ is defined recursively as $n \times (n-1)!$ if $n \geq 1$, with $0! = 1$.

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$n!$ is defined recursively as $n \times (n-1)!$ if $n \geq 1$, with $0! = 1$.

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 2 \\ 3! &= 6 \\ 4! &= 24 \\ 5! &= 120 \\ 6! &= 720 \end{aligned}$$

Factorials grow *really* quickly. $10! = 3{,}628{,}800$, $20! \approx 2.4 \times 10^{18}$, $50!$ has more than sixty digits.

Example

My hipster friend only likes 15 songs. How many different playlists can be made from these songs? Each playlist contains ten songs, and the order of the songs matters.

$$15 \times 14 \times 13 \times \cdots \times 6 = 1089728640$$

(Surprised? Permutations can be *very* large even for relatively small sets.)

The birthday paradox revisited...

What is the probability that two people in this room have the same birthday?

COMBINATIONS (UNORDERED WITHOUT REPLACEMENT)

A *combination* is a subset of X where order *doesn't* matter, A *combination* is a subset of X where order *doesn't* matter,

COMBO LUNCH MENU

Pick any three items from the following list.

Caesar Salad	Ham and Cheese	Minestrone
House Salad	Pastrami on Rye	Chicken Soup
Tuna Sandwich	Hummus Pita	Chili

How many different combo meals can I make? There are $9 \times 8 \times 7 = 504$ permutations, but this double-counts some of the meals. (Minestrone + chicken soup + chili is the same souper combo meal as chicken soup + minestrone + chili).

The question is, how many times is each combo meal counted in the full list of permutations?

This is really asking, "How many different ways can I rearrange 3 items?"

This is really asking how many permutations of 3 items can I take from a set of 3.

The answer is $P_3^3 = 3! = 6$

Therefore, each of the 504 permutations contains 6 copies of each combo meal.

Therefore, the number of combinations is $504/6 = 84$.

In general, the number of combinations is given by

$$\binom{n}{k} = \frac{P_{k,n}}{P_{k,k}} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}$$

pronounced "*n* choose *k*."

What is the relationship between $\binom{n}{k}$ and $P_{k,n}$? [1]

- A: $\binom{n}{k} \leq P_{k,n}$
- B: $\binom{n}{k} \geq P_{k,n}$
- C: $\binom{n}{k} = P_{k,n}$
- D: Impossible to say

[1] Assuming, of course, n and k are positive integers with $k \leq n$

UNORDERED SAMPLING WITH REPLACEMENT

What if order doesn't matter, but I can choose the same thing multiple times?

DELUXE COMBO LUNCH MENU

Pick any three items from the following list (repetition allowed).

Caesar Salad	Ham and Cheese	Minestrone
House Salad	Pastrami on Rye	Chicken Soup
Tuna Sandwich	Hummus Pita	Chili

Different than the previous version, I can now order 3 Caesar salads for lunch if I want.

Let $x_1, \ldots, x_9$ represent how many times I order each of the nine menu items. (So ordering 3 Caesar salads would be $x_1 = 3$, $x_2 = x_3 = \cdots = x_9 = 0$).

Then any solution to the equation $x_1 + x_2 + \cdots + x_9 = 3$ represents a possible combo meal, if each x_i is a nonnegative integer ($x_i \in \mathbb{Z}^+$).

The number of nonnegative integer solutions to this equation is the number of possible combo meals.

In total I will order three items, let me put a tick mark in each column for each order:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$\vert\vert\vert$								

I can encode this table as a string of 11 characters: 3 tick marks for the order, and 8 separators (use '$+$') between columns: |||++++++++

In fact, any string of 11 characters which contains 3 | and 8 + marks represents a valid order.

I can decide the 3 | locations (the remainder must be +), so $\binom{11}{3} = 165$.

I could just as well have chosen the 8 + locations (the rest must be |), or $\binom{11}{8} = 165$.

So, in general the formula is

$$\binom{n+k-1}{k} \quad \text{or} \quad \binom{n+k-1}{n-1}$$

EXAMPLES

Answer the following questions:

How many...

1. three-topping pizzas can be made if there are five toppings available?
2. the same as (1), but if you can repeat toppings.
3. seven-member committees can be chosen from ten eligible members?
4. four-digit PIN numbers can you make if you can't repeat digits?
5. four-digit PIN numbers can you make if you **can** repeat digits?

A more involved example

In five-card draw poker, what's the probability of being dealt a royal flush?

First, what are the outcomes and sample space?

Let A denote the event "I am dealt a royal flush." Assuming the deck is shuffled perfectly,

$$P(A) = \frac{|A|}{|S|}$$

Thus, we need to calculate $|S|$ and $|A|$.

Since the order of the cards in my hand doesn't matter in poker, the number of distinct hands I can draw is

$$\binom{52}{5} = 2598960$$

Neglecting simple rearrangements of the order of the cards, there are only four different royal flushes (one for each suit).

Therefore, the probability of a royal flush is $4/2598960 = 1/649740$.

If we wanted to use permutations, we could use the following logic:

The number of possible hands is

$$P_5^{52} = 52!/47! = 52 \times 51 \times 50 \times 49 \times 48 = 311875200$$

How many royal flushes are there? Consider each suit separately. For spades, there are exactly 5 cards I need to draw; these 5 cards can be drawn in $P_5^5 = 5! = 120$ ways.

The same is true for the other suits, so in total only 480 of the hands are royal flushes.

Therefore, the probability of a royal flush is $480/311875200 = 1/649740$.