Minimum cost flow problem

CE 377K

March 10, 2015

ANNOUNCEMENTS

HW 2 due today

Exam on Thursday, topics of coverage are through max-flow/min-cut duality (February 26 notes).

Emails marked as spam

REVIEW

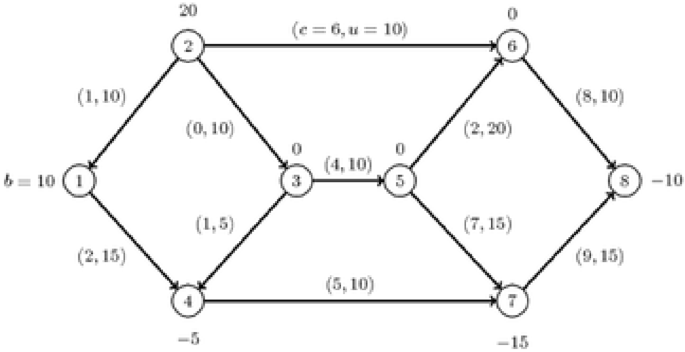
Augmenting path algorithm

Capacity scaling

Complexity arguments Proof of augmenting path algo AND max flow-min cut

MINIMUM COST FLOW PROBLEM

Minimum Cost Flow Problem



Respecting capacities, find link flows which balance supply and demand among sources and sinks, with minimum total cost.

Applications

- Logistics and shipping
- Earthwork in roadway construction
- Passenger selection in ridesharing

Each link (i, j) has a specified cost c_{ij} and capacity u_{ij}, and we must determine its flow x_{ij}.

Each node has a supply b_i. If $b_i > 0$, the node is a *source*, if $b_i < 0$ it is a *sink*, and if $b_i = 0$ it is a *transhipment* node.

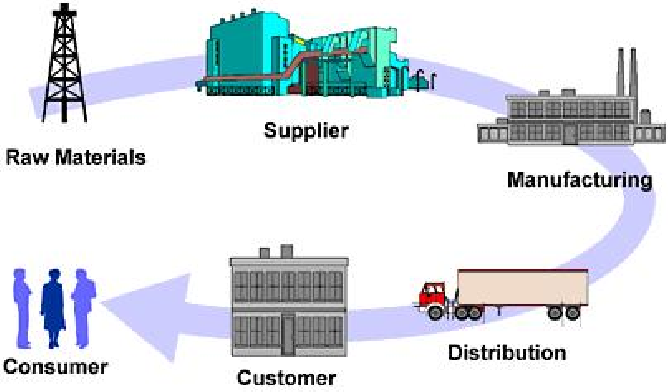
We want to minimize the total transportation cost $\sum_{(i,j)\in A} c_{ij}x_{ij}$.

Constrants are the link capacities, and flow conservation.

$$
\begin{aligned}
\max_{\mathbf{x}} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} & \\
\text{s.t.} \quad & \sum_{(i,j)\in A(i)} x_{ij} - \sum_{(h,i)\in B(i)} x_{hi} = b_i & \forall i \in N \\
& 0 \le x_{ij} \le u_{ij} & \forall (i,j) \in A
\end{aligned}
$$

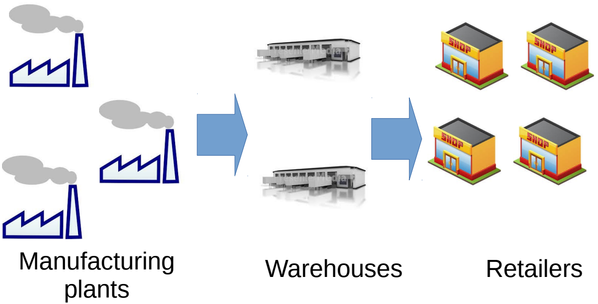
Example

Supply chain logistics can often be represented by a min cost flow problem.



The following model is based on Shahabi, Unnikrishnan, Shirazi & Boyles (2014). A three-level location-inventory problem with correlated demand. *Transportation Research Part B* 69, 1–18.

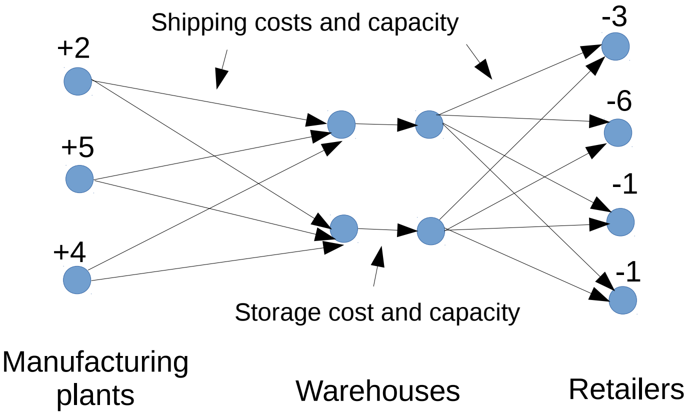
There are three kinds of nodes, representing manufacturing plants, warehouses, and retailers.
Each manufacturing plant produces a certain quantity of product, which must be shipped first to a warehouse, and then from a warehouse to a retailer which will sell a prespecified quantity of product.



(The paper considered other factors: locations of plants and warehouses were decision variables, and demand was a random variable not known in advance.)

There is a transportation cost between each plant and warehouse, and between each warehouse and retailer.

These transportation links have a capacity; furthermore, each warehouse has a capacity.



RELATION TO OTHER PROBLEMS

The minimum cost flow problem can be seen as a *generalization* of the shortest path and maximum flow problems.

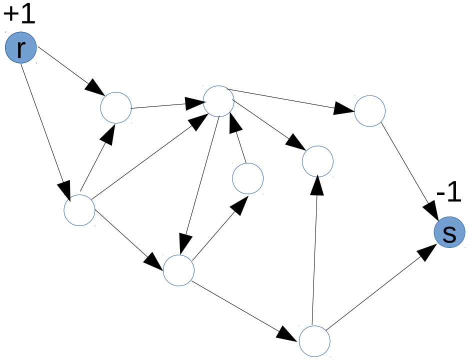
That is, by suitably choosing costs, capacities, and supplies we can solve shortest path or maximum flow using any method which will solve min cost flow.

(Naturally, this means that solving the minimum cost flow problem must be at least as hard as solving shortest path or max flow.)

Transformation to shortest path problem

For each link, set u_{ij} to ∞. (Capacities ignored in shortest path.)

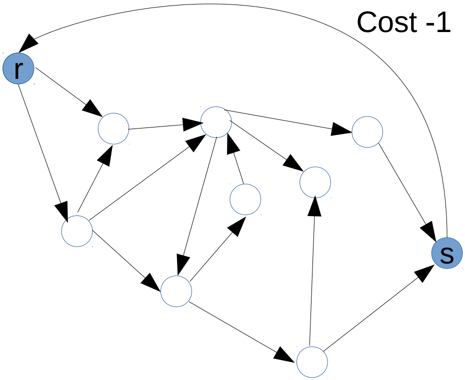
For the origin r, set $b_r = +1$. For the destination s, set $b_s = -1$. For all other nodes i, $b_i = 0$.



Transformation to maximum flow problem

Create an artificial link (s, r) connecting the sink s to the source r, with capacity $u_{sr} = \infty$.

Set $c_{ij} = 0$ for all links, except the artificial link, which gets cost $c_{sr} = -1$.



SUCCESSIVE SHORTEST PATH ALGORITHM

Like the augmenting path algorithm, the successive shortest path algorithm also uses the residual graph $\mathcal{R}(\mathbf{x})$.

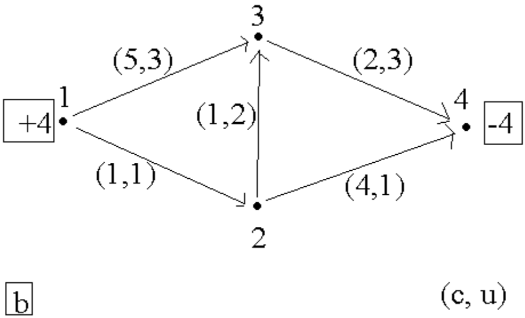
As before, the capacity of a forward link is $u_{ij} - x_{ij}$, and the capacity of a reverse link is x_{ij}.

The cost of a forward link is c_{ij}, the cost of a reverse link is $-c_{ij}$.

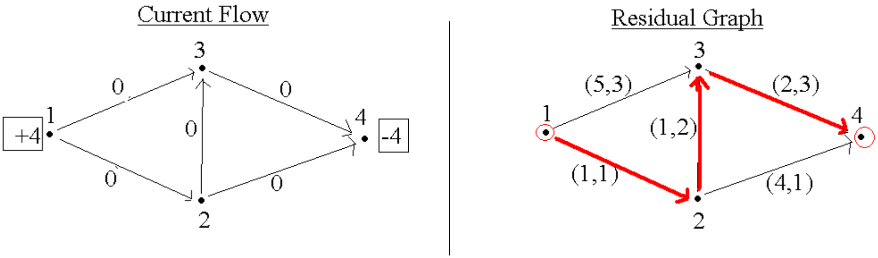
Forward links show what happens if we increase the flow on a link, reverse links show what happens if we decrease the flow on a link.

Algorithm

1. Start with the zero flow $\mathbf{x} \leftarrow \mathbf{0}$.
2. Choose a node r with $b(r) > 0$ and a sink node s with $b(s) < 0$. (If none exist, terminate.)
3. Create the residual graph $\mathbf{R}(\mathbf{x})$ and find the shortest path π from r to s with positive capacity.
4. Calculate the amount of flow which can be sent on this path: $\Delta = \min\{b(r), |b(s)|, u_{ij} : (i,j) \in \pi\}$.
5. Update the flow, adding Δ on the forward links in π and subtracting Δ from the reverse links; also reduce $b(r)$ by Δ and increase $b(s)$ by Δ.
6. Return to step 2.

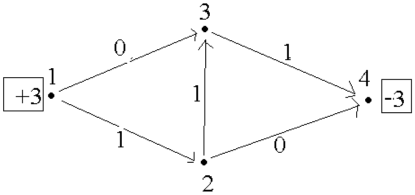

3
(5,3)
(2,3)
1
+4
(1,2)
4
-4
(1,1)
(4,1)
2
b
(c, u)

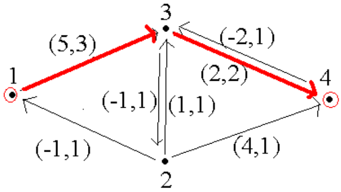
Example



(Links in residual graph labeled with cost and capacity. Zero-capacity links omitted for clarity.)

Example





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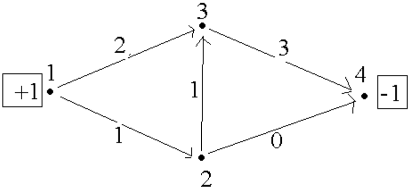

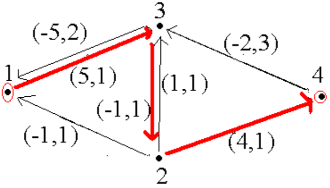

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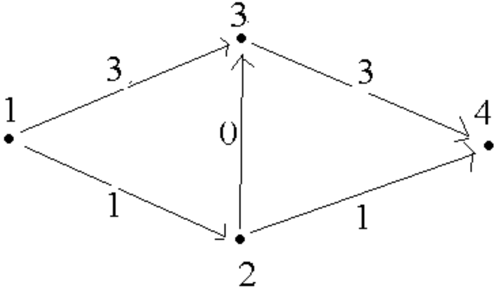
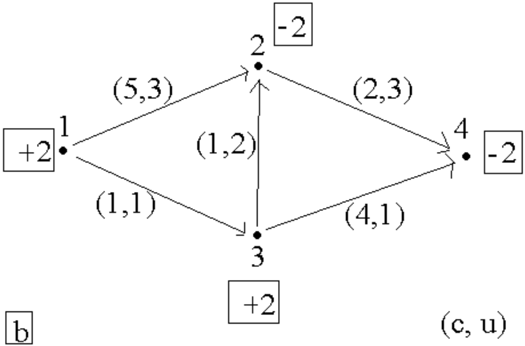
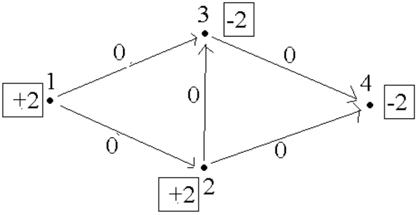
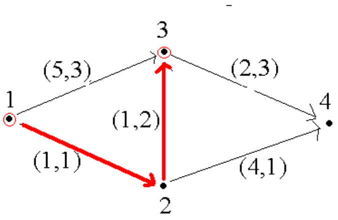


Second example



Example





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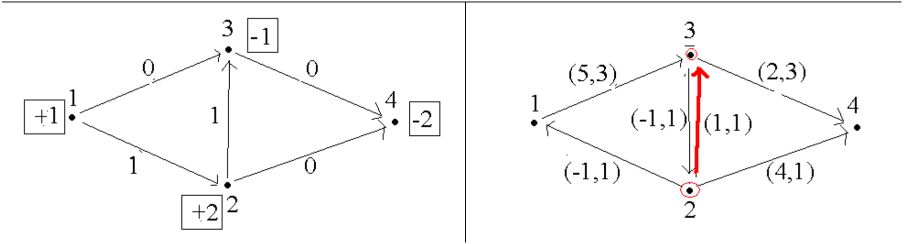

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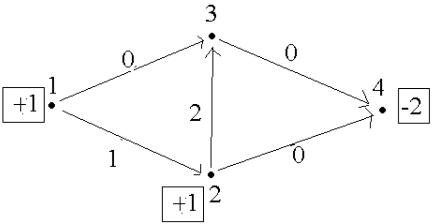

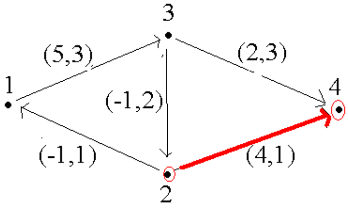

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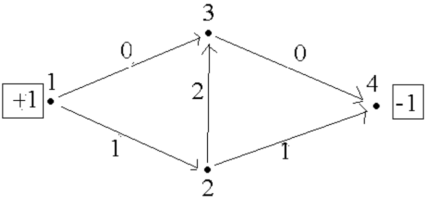

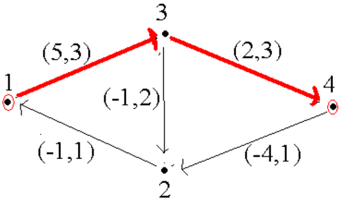

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Example