Linear programming: an introduction

CE 377K

March 24, 2015

ANNOUNCEMENTS

Exam returned today

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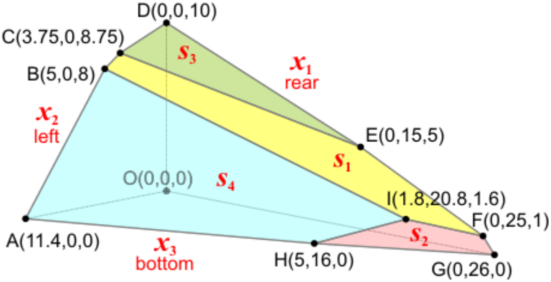
Emails marked as spam

OVERVIEW

The next few weeks we will be covering *linear* optimization problems (sometimes called linear programs.)



This is a broader class of optimization problems than network optimization.

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The objective $\min x^2$ is not.

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The constraint $x \in \{0, 0.25, 0.5, 0.75, 1\}$ is not linear.

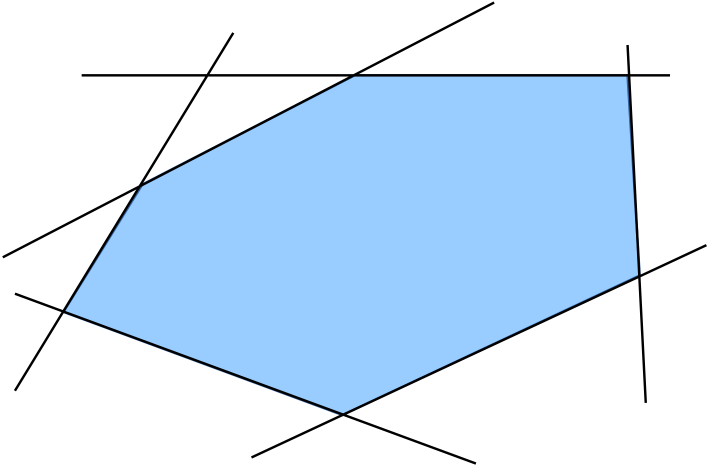
All of the network optimization problems we saw earlier in the semester are linear programs.

$$
\begin{aligned}
\max_{\mathbf{x}} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{(i,j)\in A(i)} x_{ij} - \sum_{(h,i)\in B(i)} x_{hi} = b_i && \forall i \in N \\
& 0 \le x_{ij} \le u_{ij} && \forall (i,j) \in A
\end{aligned}
$$

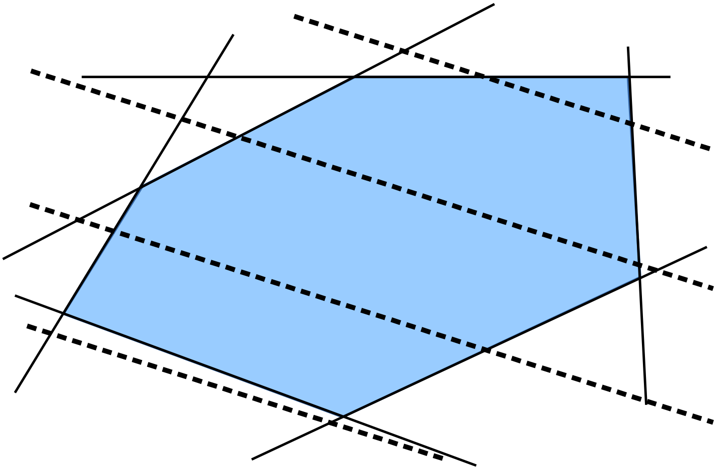
The methods discussed in the next weeks could have been used to solve shortest path, etc. But they will not be as fast as using specialized algorithms.

GEOMETRY

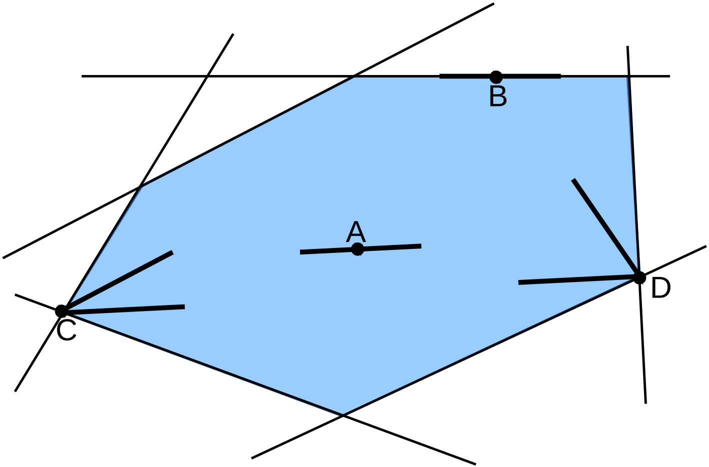
The feasible region of linear programs can be graphed as a polygon (polyhedron, in higher dimensions.)



The contours of the objective function also form lines.



An *extreme point* is a point which cannot be in the interior of any feasible line segment.



Extreme points play a very important role in linear programming.

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If a linear program has a global optimal solution and the feasible region has an extreme point, at least one of the extreme points is globally optimal.

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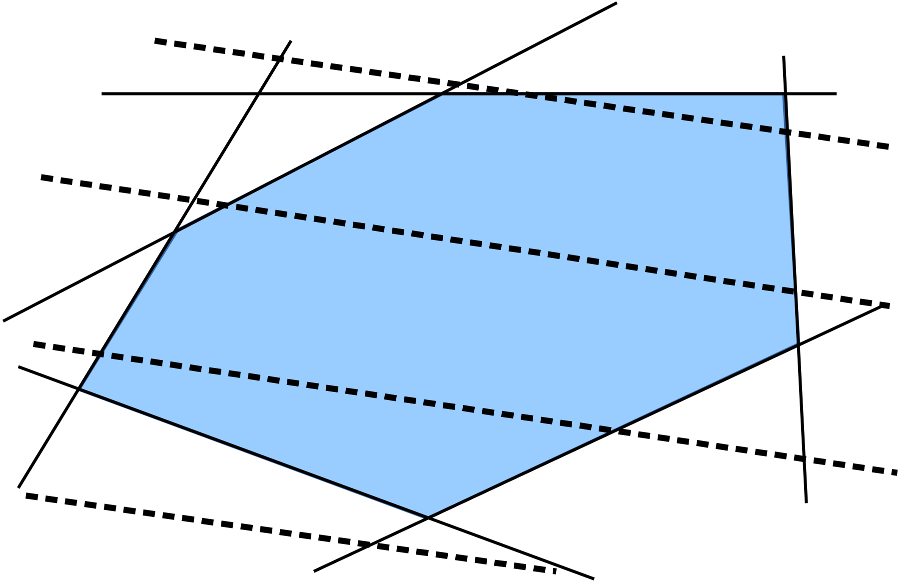
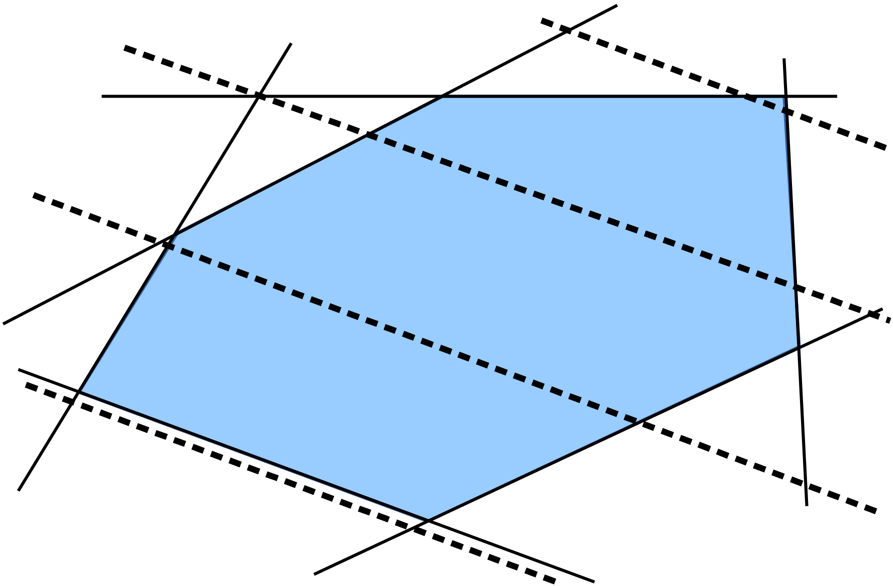
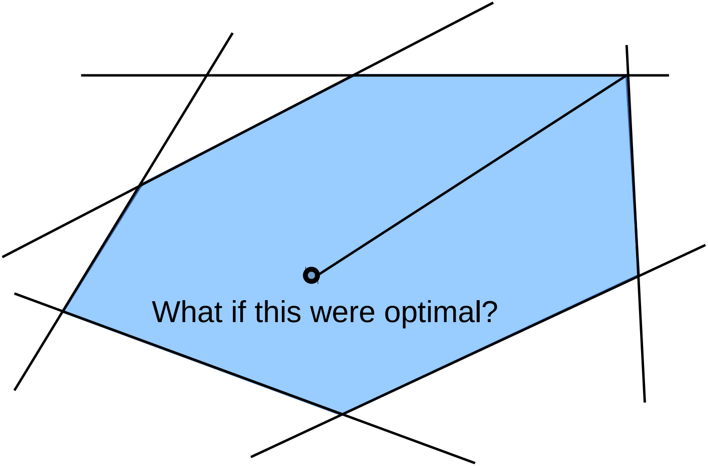
In other words, for linear programs, *we only need to look at extreme points.*

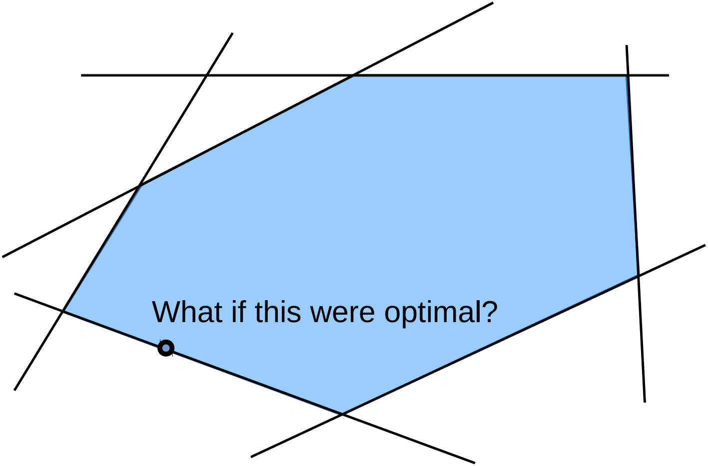




If an interior point is a global optimum, an extreme point must be a global optimum as well.



If a boundary point is a global optimum, one of the adjacent extreme points must be a global optimum as well.



EXAMPLE

A modified version of the HOT toll setting problem

The morning peak period is **two** hours long and the toll can be switched every hour. The tolls can be **any value** between \$0 and \$1, and between hours the toll cannot change by more than \$0.25. The number of people using the HOT lane between 7 and 8 AM is 2100 minus 1000 times the toll; in the second hour, the number of people using the lane is 3000 minus 1500 times the toll. Maximize the total number of travelers using the HOT lane during these three hours, without exceeding the lane capacity (2000 vehicles per hour) in either hour.

$$
\begin{aligned}
\max_{\tau_1,\tau_2} \quad & (2100 - 1000\tau_1) + (3000 - 1500\tau_2) \\
\text{s.t.} \quad & 2100 - 1000\tau_1 \leq 2000 \\
& 3000 - 1500\tau_2 \leq 2000 \\
& \tau_1 - \tau_2 \leq 0.25 \\
& \tau_2 - \tau_1 \leq 0.25 \\
& 0 \leq \tau_1, \tau_2 \leq 1
\end{aligned}
$$