# Linear programming: algebra 

## CE 377K

March 26, 2015

## ANNOUNCEMENTS

## Groups and project topics due soon

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Did everyone get my test email?

## REVIEW

Linear programming geometry

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Extreme points

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"Brute force" solution method

Linear programming geometry

Extreme points

"Brute force" solution method

More elegant solution method

## OUTLINE

- Expanding to problems with more decision variables
- Algebraic equivalent of geometric methods
- Standard form for linear programs

We are heading towards the simplex method for solving linear programs. Rather than "brute forcing" through all of the extreme points, the simplex method moves along the boundary of the feasible region in "downhill" directions.

To develop the simplex method, we need a standard form for linear programs.

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The standard form expresses a linear program in the following way:

- The objective function is to be minimized
- All decision variables are required to be non-negative
- All other constraints are equalities

$$
\begin{aligned}
\min _{x_{1}, \ldots, x_{n}} & \sum_{i=1}^{n} c_{i} x_{i} \\
\text { s.t. } & \sum_{i=1}^{n} a_{i 1} x_{i}=b_{1} \\
& \vdots \\
& \sum_{i=1}^{n} a_{i m} x_{i}=b_{m} \\
& x_{1}, \ldots, x_{n} \geq 0
\end{aligned}
$$

Sometimes we can transform a nonstandard linear program into standard form:

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If the objective is to maximize $f(x)$, we can just as well minimize $-f(x)$.

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- $x^{+} \geq 0$
- $x^{-} \geq 0$

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- $x^{+} \geq 0$
- $x^{-} \geq 0$

By substituting $x^{+}-x^{-}$everywhere $x$ appeared in the original formulation, we can express the problem in standard form.

What if there is an inequality constraint of the form $a_{1} x_{1}+a_{2} x_{2}+\cdots+a_{n} x_{n} \leq b$

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What would we do for $a \geq$ inequality constraint?

## Convert this problem into standard form

$$
\begin{array}{cl}
\max _{\tau_{1}, \tau_{2}} & \left(2100-1000 \tau_{1}\right)+\left(3000-1500 \tau_{2}\right) \\
\text { s.t. } & 2100-1000 \tau_{1} \leq 2000 \\
& 3000-1500 \tau_{2} \leq 2000 \\
& \tau_{1}-\tau_{2} \leq 0.25 \\
& \tau_{2}-\tau_{1} \leq 0.25 \\
& 0 \leq \tau_{1}, \tau_{2} \leq 1
\end{array}
$$

$$
\min _{\tau_{1}, \tau_{2}, s_{1}, s_{2}, s_{3}, s_{4}, s_{5}, s_{6}} \text { s.t. }
$$

## EXTREME POINTS ALGEBRAICALLY

When there are $n$ decision variables, an extreme point of the feasible region corresponds to the intersection of $n$ constraints (if the intersection point is feasible).

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Since the equations are linear, it is easy to find these intersection points.


Furthermore, adjacent extreme points have $n-1$ of the constraints in common.

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A preview of the simplex method: find the intersection point of $n$ constraints; then substitute constraints one at a time, moving to adjacent extreme points until the optimal solution is found.

## GENERAL ALGEBRAIC PROCESS

## Assume that a standard-form LP has $n$ decision variables and $m$ equality constraints.

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It is safe to assume that $n \geq m$. (Why?)

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Any feasible point must satisfy all $m$ equality constraints, so $n-m$ of the nonnegativity constraints must be tight.

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Any feasible point must satisfy all $m$ equality constraints, so $n-m$ of the nonnegativity constraints must be tight.

That is, $n-m$ of the decision variables must be zero, and the other $m$ can take positive or zero values.

The $m$ decision variables which take take positive or zero values are called basic variables or a basis, and the remaining $n-m$ decision variables are called nonbasic.

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We can identify the extreme point corresponding to a basis by solving $m$ equations in $m$ unknowns.

$$
\begin{array}{rl}
\min _{\tau_{1}, \tau_{2}, s_{1}, s_{2}, s_{3}, s_{4}, s_{5}, s_{6}} & 1000 \tau_{1}+1500 \tau_{2}-5100 \\
\text { s.t. } & 2100-1000 \tau_{1}+s_{1}=2000 \\
& 3000-1500 \tau_{2}+s_{2}=2000 \\
& \tau_{1}-\tau_{2}+s_{3}=0.25 \\
& \tau_{2}-\tau_{1}+s_{4}=0.25 \\
& \tau_{1}+s_{5}=1 \\
& \tau_{2}+s_{6}=1 \\
& \tau_{1}, \tau_{2}, s_{1}, s_{2}, s_{3}, s_{4}, s_{5}, s_{6} \geq 0
\end{array}
$$

Choose $\tau_{1}, \tau_{2}, s_{1}, s_{2}, s_{3}, s_{4}$ as basic variables.

A basis is feasible if the solution to these $m$ equations consists of nonnegative values.

$$
\begin{array}{rl}
\min _{\tau_{1}, \tau_{2}, s_{1}, s_{2}, s_{3}, s_{4}, s_{5}, s_{6}} & 1000 \tau_{1}+1500 \tau_{2}-5100 \\
\text { s.t. } & 2100-1000 \tau_{1}+s_{1}=2000 \\
& 3000-1500 \tau_{2}+s_{2}=2000 \\
& \tau_{1}-\tau_{2}+s_{3}=0.25 \\
& \tau_{2}-\tau_{1}+s_{4}=0.25 \\
& \tau_{1}+s_{5}=1 \\
& \tau_{2}+s_{6}=1 \\
& \tau_{1}, \tau_{2}, s_{1}, s_{2}, s_{3}, s_{4}, s_{5}, s_{6} \geq 0
\end{array}
$$

Choose $s_{1}, s_{2}, s_{3}, s_{4}, s_{5}, s_{6}$ as basic variables.

