Simplex tableau

CE 377K

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Review

Reduced costs

Basic and nonbasic variables

OUTLINE

- Review by example: simplex method demonstration
- Simplex tableau

Example

You own a small firm producing construction materials for the EERC building. Your firm produces three materials: concrete, mortar, and grout. Producing one ton of each these materials requires the use of a limited resource, as shown in the following table:

Resource	Concrete	Mortar	Grout	Availability
Labor	1 hour	2 hours	2 hours	20 hours
Blending time	2 hours	1 hour	2 hours	20 hours
Storage capacity	2 m^3	2 m^3	1 m^3	20 m^3
Sales price	\$20	\$30	\$40	
Manufacturing cost	\$10	\$18	\$28	

What is your optimal production strategy?

A first attempt at writing down the optimization problem is:

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & 10x_1 + 12x_2 + 12x_3 & \\ \text{s.t.} \quad & x_1 + 2x_2 + 2x_3 & \leq 20 \\ & 2x_1 + x_2 + 2x_3 & \leq 20 \\ & 2x_1 + 2x_2 + x_3 & \leq 20 \\ & x_1, x_2, x_3 & \geq 0 \end{aligned}$$

After putting the problem in standard form we have:

$$\begin{aligned}
\min_{x_1,\dots,x_6} \quad & -10x_1 - 12x_2 - 12x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\
& 2x_1 + x_2 + 2x_3 + x_5 &= 20 \\
& 2x_1 + 2x_2 + x_3 + x_6 &= 20 \\
& x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}$$

Simplex method

1. Identify an initial feasible basis $\{x_1^B, \dots, x_m^B\}$.
2. Calculate the reduced cost $\overline{c^k} = c^k - \mathbf{c^B B^{-1} A^k}$ for each nonbasic decision variable.
3. If all of the reduced costs are nonnegative, the current basis is optimal. **STOP.**
4. Otherwise, choose some nonbasic decision variable x^k with a negative reduced cost to enter the basis.
5. Set $x^k = \min_{i:\Delta x_i < 0} x_i / |\Delta x_i|$; the basic variable leaving the basis is one where this minimum is obtained.
6. Return to step 2.

By inspection, a feasible basis is $\{x_4, x_5, x_6\}$, which corresponds to the the extreme point $(0, 0, 0, 20, 20, 20)$. We have

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore $\mathbf{B}^{-1} = \mathbf{B}$ and the reduced cost vector is

$$\bar{\mathbf{c}} = \mathbf{c} - \mathbf{c}^{\mathbf{B}}\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} -10 & -12 & -12 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -10 & -10 & -12 & 0 & 0 & 0 \end{bmatrix}$$

The reduced cost of x_1 is negative, so we choose it to enter the basis. We calculate the change in the existing basic variables:

$$\Delta \mathbf{x}^{\mathbf{B}} = \mathbf{B}^{-1}\mathbf{A}^{1} = \mathbf{A}^{1} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Checking the ratio test, we see that $x_4/\Delta x_4 = 20$, $x_5/\Delta x_5 = 10$, and $x_6/\Delta x_6 = 10$, so one of x_5 and x_6 must leave the basis; let's assume that x_5 leaves, so the new basis is $\{x_4, x_1, x_6\}$. Now

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 0.5 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

The current solution is $\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 10 & 0 & 0 & 10 & 0 & 0 \end{bmatrix}$ and

$$\bar{\mathbf{c}} = \begin{bmatrix} -10 & -12 & -12 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -10 & 0 \end{bmatrix} \begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 0.5 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -7 & -2 & 0 & 5 & 0 \end{bmatrix}$$

The reduced cost of x_3 is negative, so we choose it to enter the basis. We calculate

$$\Delta \mathbf{x}^{\mathbf{B}} = \mathbf{B}^{-}\mathbf{1}A^3 = \begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 0.5 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Checking the ratio test, we see that $x_4/\Delta x_4 = 10$ and $x_1/\Delta x_1 = 10$, so one of x_4 and x_1 must leave the basis; let's assume that x_4 leaves, so the new basis is $\{x_3, x_1, x_6\}$.

Now

$$\mathbf{B} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1 & -0.5 & 0 \\ -1 & 1 & 0 \\ 1 & -1.5 & 1 \end{bmatrix}$$

so the current solution is $\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 0 & 0 & 10 & 0 & 0 & 10 \end{bmatrix}$ and

$$\begin{aligned} \bar{\mathbf{c}} = \begin{bmatrix} -10 & -12 & -12 & 0 & 0 & 0 \end{bmatrix} - \\ \begin{bmatrix} -12 & -10 & 0 \end{bmatrix} \begin{bmatrix} 1 & -0.5 & 0 \\ -1 & 1 & 0 \\ 1 & -1.5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 0 & -4 & 0 & 2 & 4 & 0 \end{bmatrix} \end{aligned}$$

The reduced cost of x_2 is negative, so we choose it to enter the basis. We calculate

$$\Delta \mathbf{x}^{\mathbf{B}} = \mathbf{B}^{-1} A^2 = \begin{bmatrix} 1 & -0.5 & 0 \\ -1 & 1 & 0 \\ 1 & -1.5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -1 \\ 2.5 \end{bmatrix}$$

Checking the ratio test, we see that $x_3/\Delta x_3 = 6.67$ and $x_6/\Delta x_6 = 4$, so x_6 must leave the basis, and the new basis is $\{x_3, x_1, x_2\}$.

Now

$$\mathbf{B} = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \qquad \mathbf{B}^{-1} = \begin{bmatrix} 0.4 & 0.4 & -0.6 \\ -0.6 & 0.4 & 0.4 \\ 0.4 & -0.6 & 0.4 \end{bmatrix}$$

so the current solution is $\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 4 & 4 & 4 & 0 & 0 & 0 \end{bmatrix}$ and

$$\begin{aligned} \bar{\mathbf{c}} = \begin{bmatrix} -10 & -12 & -12 & 0 & 0 & 0 \end{bmatrix} - \\ \begin{bmatrix} -12 & -10 & -12 \end{bmatrix} \begin{bmatrix} 0.4 & 0.4 & -0.6 \\ -0.6 & 0.4 & 0.4 \\ 0.4 & -0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 & 0 & 3.6 & 1.6 & 1.6 \end{bmatrix} \end{aligned}$$

All reduced costs are nonnegative, so the current basis is optimal.

The optimal solution is $\mathbf{x}^* = \begin{bmatrix} 4 & 4 & 4 & 0 & 0 & 0 \end{bmatrix}$, and the optimal objective function value is $\mathbf{c} \cdot \mathbf{x}^* = -136$.

SIMPLEX TABLEAU

The *simplex tableau* is a more convenient way to perform the computations needed by the simplex method.

The tableau is a table containing the following elements:

$-Z$	$\dots$	$c^k - \mathbf{c^B} \mathbf{B}^{-1} \mathbf{A^k}$	$\dots$
$\vdots$			
x_i^B	$\dots$	$\mathbf{B}^{-1} \mathbf{A^k}$	$\dots$
$\vdots$			

$-Z$	$\ldots \quad c^k - \mathbf{c^B B^{-1} A^k} \quad \ldots$
$\vdots$ x_i^B $\vdots$	$\ldots \quad \mathbf{B^{-1} A^k} \quad \ldots$

1. There are $n + 1$ columns and $m + 1$ rows, numbered starting from zero.
2. $\mathcal{T}_{ij}$ is the value in the i-th row and the j-th column.
3. $\mathcal{T}_{00}$ is the *negative* of the objective function value.
4. $\mathcal{T}_{0j}$ is the reduced cost of x_j.
5. $\mathcal{T}_{i0}$ has the current value of the i-th basic variable.
6. The other entries $\mathcal{T}_{ij}$ show the rate by which x_i would *decrease* if x_j is added to the basis.

After finding an initial feasible basis, all remaining steps of the simplex method can be computed using the tableau.

1. Reduced costs for all variables are already shown in the first row.
2. We can see by inspection whether reduced costs are all nonnegative, in which case we are done.
3. If not, we can visually pick one of the x_k with negative reduced cost to enter the basis.
4. To see which variable leaves the basis, compare the ratio of the values in the zero-th column to the k-th column.
5. Changing the basis, updating $\mathbf{B}^{-1}\mathbf{A}$ and reduced costs.

All steps but the last are easy.

The initial tableau corresponding to the example problem is:

0	-10	-12	-12	0	0	0
20	1	2	2	1	0	0
20	2	1	2	0	1	0
20	2	2	1	0	0	0

By choosing x_4, x_5, x_6 as the initial basis, $\mathbf{B}^{-1} = \mathbf{I}$ and $\mathbf{c}^{\mathbf{B}} = \mathbf{0}$ so the tableau was easy to write down. Next class we'll see how we can start off in general.

0	-10	-12	-12	0	0	0
20	1	2	2	1	0	0
20	2	1	2	0	1	0
20	2	2	1	0	0	0

The reduced cost of x_1 is negative, so choose it to enter the basis.

0	-10	-12	-12	0	0	0
20	1	2	2	1	0	0
20	2	1	2	0	1	0
20	2	2	1	0	0	0

In the ratio test, $\mathcal{T}_{10}/\mathcal{T}_{11} = 20$, $\mathcal{T}_{20}/\mathcal{T}_{21} = 10$, and $\mathcal{T}_{30}/\mathcal{T}_{31} = 10$ so we could choose either x_5 or x_6 to leave the basis.

Since x_1 is entering the basis and x_5 is leaving, we want to make column x_1 of the tableau look like x_5 does currently.

We accomplish this with the following matrix row manipulations (performed in this order):

- Divide every element in the second row by 2.
- Add ten times the second row to the zero-th row.
- Subtract the second row from the first row.
- Subtract twice the second row from the third row.

0	-10	-12	-12	0	0	0
20	1	2	2	1	0	0
10	1	0.5	1	0	0.5	0
20	2	2	1	0	0	0

100	0	-7	-2	0	5	0
10	0	1.5	1	1	-0.5	0
10	1	0.5	1	0	0.5	0
0	0	1	-1	0	-1	1

This is the end of the first iteration.

In general, to change the basis, assume that x_k is entering the basis, and that x_ℓ is leaving.

1. Divide every element in the ℓ-th row by $\mathcal{T}_{\ell k}$. This sets $\mathcal{T}_{\ell k} = 1$
2. For each of the other rows i, multiply the ℓ-th row by $\mathcal{T}_{ik}$ and subtract it from the i-th row. This sets $\mathcal{T}_{ik} = 0$ for all $i \neq \ell$.

Here is the tableau at the end of the first iteration:

100	0	-7	-2	0	5	0
10	0	1.5	1	1	-0.5	0
10	1	0.5	1	0	0.5	0
0	0	1	-1	0	-1	1

1. The basic variables correspond to the columns which form the identity matrix (x_4, x_1, x_6)
2. The reduced costs of x_2 and x_3 are still negative, so we can choose one of them to enter the basis.

Say we chose x_3 to enter the basis. Based on the ratio test, we force x_4 (row 1) to leave the basis.

100	0	-7	-2	0	5	0
10	0	1.5	1	1	-0.5	0
10	1	0.5	1	0	0.5	0
0	0	1	-1	0	-1	1

1. Divide the first row by 1 (no change)
2. Add twice the first row to the zero-th row
3. Subtract the first row from the second row
4. Add the first row to the third row

Here is the tableau at the end of the second iteration:

120	0	-4	0	2	4	0
10	0	1.5	1	1	-0.5	0
0	1	-1	0	-1	1	0
10	0	2.5	0	1	-1.5	1

1. The basic variables correspond to the columns which form the identity matrix (x_3, x_1, x_6)
2. The reduced cost of x_2 is still negative, so it enters the basis.

Since x_2 is entering the basis, the ratio test forces x_6 (row 3) to leave the basis.

120	0	-4	0	2	4	0
10	0	1.5	1	1	-0.5	0
0	1	-1	0	-1	1	0
10	0	2.5	0	1	-1.5	1

1. Divide the third row by 2.5
2. Add four times the third row to the zero-th row.
3. Subtract 1.5 times the third row to the first row.
4. Add the third row to the second row.

Here is the tableau at the end of the third iteration:

136	0	0	0	3.6	1.6	1.6
4	0	0	1	0.4	0.4	-0.6
4	1	0	0	-0.6	0.4	0.4
4	0	1	0	0.4	-0.6	0.4

All reduced costs are nonnegative, so this is the optimal solution. The basis is $x_3 = 4$, $x_1 = 4$, $x_2 = 4$ and the objective function value is -136.