Linear programming odds and ends

CE 377K

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Review

Standard form (in matrix notation)

Reduced costs

Simplex method

Simplex tableau

OUTLINE

- Finding a feasible solution (big M method)
- Sensitivity analysis (to changes in $\mathbf{c}$ or $\mathbf{b}$)

BIG M METHOD

With our simplex method examples so far, it has been easy to find a feasible basis.

It is not always obvious how to do this. The big M method is a technique for starting the simplex method.

Our process was easy because we found an identity matrix inside $\mathbf{A}$ which could serve as the initial basis.

If there are m constrains, introduce m new "artificial" decision variables $y_1, \cdots, y_m$. (You do not need to introduce an artificial variable if a column in $\mathbf{A}$ is already a column from the identity matrix.)

We want to add these variables into the optimization problem (objective and constraints) in such a way that:

1. It is easy to find an initial feasible solution to the problem.
2. The presence of the artificial variables does not affect the optimal solution.

Think of the artificial variables as scaffolding. They help the construction process but are not part of the final building.

For the m-th constraint, add $+y_m$. (This will form an identity matrix in the columns corresponding to these variables.)

The objective function is changed by giving each y_i a coefficient of $+M$ (where M is a “big number”)

$$
\begin{aligned}
\min_{x_1,\dots,x_4} \quad & x_1 + x_2 + x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 + 3x_3 && = 3 \\
& -x_1 + 2x_2 + 6x_3 && = 2 \\
& 4x_2 + 9x_3 && = 5 \\
& 3x_3 + x_4 && = 1 \\
& x_1, x_2, x_3, x_4 && \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min_{x_1,\ldots,x_4,y_1,\ldots,y_3} \quad & x_1 + x_2 + x_3 + My_1 + My_2 + My_3 \\
\text{s.t.} \quad & x_1 + 2x_2 + 3x_3 + y_1 && = 3 \\
& -x_1 + 2x_2 + 6x_3 + y_2 && = 2 \\
& 4x_2 + 9x_3 + y_3 && = 5 \\
& 3x_3 + x_4 && = 1 \\
& x_1, x_2, x_3, x_4, y_1, y_2, y_3 && \geq 0
\end{aligned}
$$

Example

$$
\begin{aligned}
\min_{x_1,\ldots,x_3,y_1,\ldots,y_4} \quad & x_1 + x_2 + x_3 + My_1 + My_2 + My_3 \\
\text{s.t.} \quad & x_1 + 2x_2 + 3x_3 + y_1 && = 3 \\
& -x_1 + 2x_2 + 6x_3 + y_2 && = 2 \\
& 4x_2 + 9x_3 + y_3 && = 5 \\
& 3x_3 + y_4 && = 1 \\
& x_1, x_2, x_3, y_1, y_2, y_3 && \geq 0
\end{aligned}
$$

With this problem, it is easy to find an initial solution: choose $y_1, \ldots, y_3, x_4$ as our basis.

Example

$$
\begin{aligned}
\min_{x_1,\ldots,x_3,y_1,\ldots,y_4} \quad & x_1 + x_2 + x_3 + My_1 + My_2 + My_3 \\
\text{s.t.} \quad & x_1 + 2x_2 + 3x_3 + y_1 && = 3 \\
& -x_1 + 2x_2 + 6x_3 + y_2 && = 2 \\
& 4x_2 + 9x_3 + y_3 && = 5 \\
& 3x_3 + y_4 && = 1 \\
& x_1, x_2, x_3, y_1, y_2, y_3 && \geq 0
\end{aligned}
$$

Furthermore, since M is a "big number," the optimal solution will have all $y_i = 0$ (unless the original problem is infeasible).

To start the simplex method, we calculate the initial reduced costs:

$$\bar{\mathbf{c}} = \mathbf{c} - \mathbf{c}^{\mathbf{B}}\mathbf{B}^{-1}\mathbf{A} =$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & M & M & M \end{bmatrix} - \begin{bmatrix} M & M & M & 0 \end{bmatrix} \mathbf{I} \begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ -1 & 2 & 6 & 0 & 0 & 1 & 0 \\ 0 & 4 & 9 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -8M+1 & -18M+1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From here, we can run the simplex method as before:

$-10M$	1	$-8M+1$	$-18M+1$	0	0	0	0
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

x_3 enters the basis, x_4 leaves

$-4M - 1/3$	1	$-8M + 1$	0	$6M - 1/3$	0	0	0
2	1	2	0	-1	1	0	0
0	-1	2	0	-2	0	1	0
2	0	4	0	-3	0	0	1
$1/3$	0	0	1	$1/3$	0	0	0

x_2 enters the basis, y_2 leaves the basis.

$-4M-1/3$	$-4M+3/2$	0	0	$-2M+2/3$	0	$4M-1/2$	0
2	2	0	0	-1	1	-1	0
0	$-1/2$	1	0	-1	0	$1/2$	0
2	2	0	0	1	0	-2	1
$1/3$	0	0	1	$1/3$	0	0	0

x_1 enters the basis, y_1 leaves the basis.

$-11/6$	0	0	0	$-1/12$	$2M - 3/4$	$2M + 1/4$	0
1	1	0	0	$1/2$	$1/2$	$-1/2$	0
$1/2$	0	1	0	$-3/4$	$1/4$	$1/4$	0
0	0	0	0	0	-1	-1	1
$1/3$	0	0	1	$1/3$	0	0	0

x_4 enters the basis, x_3 leaves the basis.

$-7/4$	0	0	$1/4$	0	$2M - 3/4$	$2M + 1/4$	0
$1/2$	1	0	$-3/2$	0	$1/2$	$-1/2$	0
$5/4$	0	1	$9/4$	0	$1/4$	$1/4$	0
0	0	0	0	0	-1	-1	1
1	0	0	3	1	0	0	0

All reduced costs are nonnegative, so we're done. The optimal solution is $x_1 = 1/2$, $x_2 = 5/4$, $x_7 = 0$, $x_4 = 1$, all other decision variables equal to zero.

What would have happened to the method if the right-hand side of one of the constraints was negative? Could we fix this?

SIMPLEX METHOD IN PRACTICE

We have not yet talked about the computational complexity of the simplex method.

In the worst case, the simplex method can require exponentially many steps.

However, for practical problems it is usually very fast.

This is one place where big O notation can be misleading, since it is a “worst case” performance bound.

There are other algorithms which have been developed for solving linear programs:

- The *ellipsoid method* has polynomial worst-case complexity, but is actually very slow in practice.
- The *interior point method* is worst-case polynomial, and is comparable to the simplex method for practical problems.

SENSITIVITY ANALYSIS OF LINEAR PROGRAMS

Often the “input data” for the problem is not known exactly, or may change without warning.

In particular, we might ask the following questions?

- What if one of the requirements b_j changes?
- What if one of the objective function coefficients c_i changes?
- What if the structure of the constraints a_{ij} changes?

Assume we know the optimal solution for some $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$; what happens if one of these changes?

The results of the simplex method make it easy to see some of these changes. In this class we'll focus on changes to the requirement vector b_i, and changes to the objective coefficients c_i.

If there is a change to the problem, there are two possible concerns:

- Is the previous optimal basis still *feasible*?
- Is the previous optimal basis still *optimal*?

Change to the requirement vector

Assume that the j-th component of $\mathbf{b}$ changes from b_j to $b'_j = b_j + \delta$.

If the previous optimal basis is feasible, then it is still optimal.

Why? The reduced costs are $\overline{c}^k = c^k - \mathbf{c^B B^{-1} A^k}$.

These do not depend on $\mathbf{b}$, so if the reduced costs were nonnegative with $\mathbf{b}$, they will also be nonnegative under $\mathbf{b}'$.

However, if δ is large enough, the old basis is no longer feasible (and thus not optimal).

We have a formula for the new basic variables: $\mathbf{x^B} = \mathbf{B}^{-1}\mathbf{b}$ so

$$(\mathbf{x^B})' = \mathbf{B}^{-1}\mathbf{b}' = \mathbf{x^B} + \delta \mathbf{B}^{-1}\mathbf{e_j}$$

where $\mathbf{e_j}$ is a column vector of all zeroes (except for a value of 1 in the j-th row)

Since $(\mathbf{x^B})' = \mathbf{B^{-1}b'} = \mathbf{x^B} + \delta\mathbf{B^{-1}e_j}$, we know that the old optimal solution $\mathbf{x^B}$ is perturbed by $\delta\mathbf{B^{-1}e_j}$.

Remember the bottom right portion of the tableau represents $\mathbf{B^{-1}A}$.

So, if one of the columns of the constraint matrix is e_j, we just need to look at that column of the final tableau.

If we were using the big-M method, *there will always be such a column.*

So, if $(\mathbf{x^B})' = \mathbf{x^B} + \delta \mathbf{B^{-1} e_j} \geq \mathbf{0}$, then the old basis is still feasible.

This vector equation breaks down into m scalar equations of the form

$$x_i + \delta \mathcal{T}_{ik} \geq 0$$

This means that if $\mathcal{T}_{ik} > 0$, we need $\delta \geq -x_i / \mathcal{T}_{ik}$ for the current basis to remain feasible.

If $\mathcal{T}_{ik} < 0$, we need $\delta \leq -x_i / \mathcal{T}_{ik}$.

Since *each* of the m constraints imposes such a restriction on δ, we have that the current basis remains optimal iff

$$\max_{i:\mathcal{T}_{ik}>0} -\frac{x_i}{\mathcal{T}_{ik}} \leq \delta \leq \min_{i:\mathcal{T}_{ik}<0} -\frac{x_i}{\mathcal{T}_{ik}}$$

Example

The problem

$$\begin{aligned} \min_{x_1,x_2,x_3,x_4} \quad & -5x_1 - x_2 + 12x_3 & \\ \text{s.t.} \quad & 3x_1 + 2x_2 + x_3 & = 10 \\ & 5x_1 + 3x_2 + x_4 & = 16 \\ & x_1, x_2, x_3, x_4 & \geq 0 \end{aligned}$$

has optimal tableau

12	0	0	2	7
2	1	0	−3	2
2	0	1	5	−3

What is the optimal solution? How much can the right-hand side of the first constraint change without affecting the optimal basis?

12	0	0	2	7
2	1	0	-3	2
2	0	1	5	-3

Since the third column of $\mathbf{A}$ was $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the third column of the tableau gives us the information we need.

If $\delta \in [-2/5, 2/3]$ the current basis remains feasible.

Change to the cost vector $\mathbf{c}$

Now, assume that the requirement vector $\mathbf{b}$ is the same, but one of the cost coefficients is changed from c_i to $c_i' = c_i + \delta$

Here, there are no concerns about feasibility since none of the constraints have changed.

However, the old solution may no longer be optimal.

The reduced costs are the key to checking optimality.

As before, assume that we have the optimal solution (and optimal tableau) for the original optimization problem.

The old basis remains optimal as long as all of the reduced costs remain nonnegative.

So, we need

$$\overline{\mathbf{c}}' = \mathbf{c}' - \mathbf{c^{B}}'\mathbf{B}^{-1}\mathbf{A} \geq 0$$

which is equivalent to n scalar equations of the form $c_k' - \mathbf{c^{B}}'\mathbf{B}^{-1}\mathbf{A^k} \geq 0$

The easy case: assume that x_i is nonbasic. Then $\mathbf{c^{B'}} = \mathbf{c^B}$, and all of the equations stay the same except for the one involving x_i.

This one is simply $\mathbf{c^B B^{-1} A^i} \leq c^i + \delta$ or $\delta \geq -\overline{c}_i = -\mathcal{T}_{0i}$ which is easily obtained from the final tableau.

Example

The problem

$$\begin{aligned}
\min_{x_1,x_2,x_3,x_4} \quad & -5x_1 - x_2 + 12x_3 \\
\text{s.t.} \quad & 3x_1 + 2x_2 + x_3 && = 10 \\
& 5x_1 + 3x_2 + x_4 && = 16 \\
& x_1, x_2, x_3, x_4 && \geq 0
\end{aligned}$$

has optimal tableau

12	0	0	2	7
2	1	0	−3	2
2	0	1	5	−3

How much can x_3 change for the current solution to remain optimal?

12	0	0	2	7
2	1	0	-3	2
2	0	1	5	-3

Since x_3 is nonbasic, we simply need $\delta \geq -\overline{c}_3 = -2$. For any such δ the current solution remains optimal.

The slightly harder case is if x_i is basic (say it is the ℓ-th basic variable). In this case $\mathbf{c^B}$ changes as well as $\mathbf{c}$.

So, the new reduced cost equation is $\mathbf{c^B B^{-1} A} + \delta \mathbf{e}_\ell \mathbf{B^{-1} A} \leq \mathbf{c} + \mathbf{e_i}$ which simplifies to

$$\delta[\mathbf{B^{-1} A^j}]_\ell \leq c_j - \mathbf{c^B B^{-1} A} = \overline{c}_j$$

or, from the tableau,

$$\mathcal{T}_{\ell i}\delta \leq \overline{c}_j$$

for all j.

As we did for changes in the requirement vector, this last formula simplifies to

$$\max_{j \neq i : \mathcal{T}_{\ell j} < 0} \left\{ \frac{\overline{c}_j}{\mathcal{T}_{\ell j}} \right\} \leq \delta \leq \min_{j \neq i : \mathcal{T}_{\ell j} > 0} \left\{ \frac{\overline{c}_j}{\mathcal{T}_{\ell j}} \right\}$$

Example

The problem

$$\begin{aligned} \min_{x_1,x_2,x_3,x_4} \quad & -5x_1 - x_2 + 12x_3 & \\ \text{s.t.} \quad & 3x_1 + 2x_2 + x_3 & = 10 \\ & 5x_1 + 3x_2 + x_4 & = 16 \\ & x_1, x_2, x_3, x_4 & \geq 0 \end{aligned}$$

has optimal tableau

$$\begin{array}{|c|cccc|} \hline 12 & 0 & 0 & 2 & 7 \\ \hline 2 & 1 & 0 & -3 & 2 \\ 2 & 0 & 1 & 5 & -3 \\ \hline \end{array}$$

How much can x_1 change for the current solution to remain optimal?

12	0	0	2	7
2	1	0	-3	2
2	0	1	5	-3

Since x_1 is basic, we need to compare the ratios of reduced costs for each nonbasic variable to the tableau entry: $\delta \in [-2/3, 7/2]$ for the current basis to remain optimal.