Examples of Optimization Problems

CE 377K

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REVIEW

Objective function, decision variables, constraints...

Transit frequency setting problem

Notation

Feasible solutions and set

MORE EXAMPLES OF FORMULATIONS

Maintenance scheduling

You must schedule routine maintenance on a set of pavement sections over the next 10 years. Each section can be described by a condition index from 0 to 100. Each section deteriorates at a known, constant rate, but if you perform maintenance in a given year, its condition will improve by a constant ammount. Given a budget for each year, when and where should you perform maintenance to maximize the average condition?

Let the decision variable $x_f^t = 1$ if maintenance is performed on facility f during year t, and 0 otherwise.

Let k_f be the cost of maintenance on facility f, and B^t the budget in year t.

There are 10 budget constraints, of the form $\sum_{f \in F} k_f x_f^t \leq B^t$

Let the condition of facility f at the end of year t be c_f^t, let d_f be the annual deterioration rate, and i_f the improvement if maintenance is performed. *(Which of these are decision variables?)*

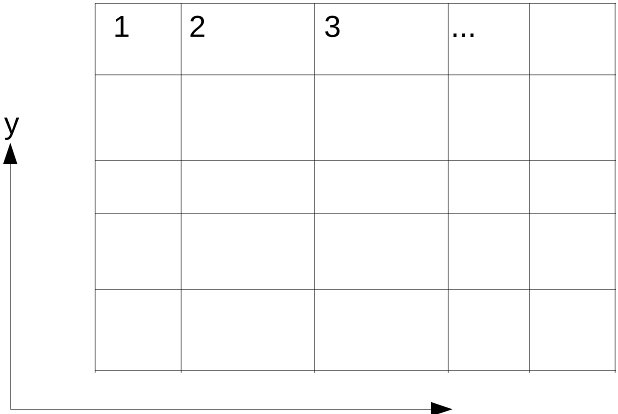
Then $c_f^t = c_f^{t-1} - d_f + x_f^t i_f$, plus the requirement that the condition be between 0 and 100.

What is the objective function?

$$
\begin{aligned}
\min_{\mathbf{x},\mathbf{c}} \quad & \frac{1}{10|F|}\sum_{f\in F}\sum_{t=1}^{10} c_f^t \\
\text{s.t.} \quad & \sum_{f\in F} k_f x_f^t \le B^t && \forall t \in \{1,\dots 10\} \\
& c_f^t = \begin{cases} 100 & \text{if } c_f^{t-1} - d_f + x_f^t i_f > 100 \\ 0 & \text{if } c_f^{t-1} - d_f + x_f^t i_f < 0 \\ c_f^{t-1} - d_f + x_f^t i_f & \text{otherwise} \end{cases} && \forall f \in F, t \in \{1,2,\dots,10\} \\
& x_f^t \in \{0,1\} && \forall f \in F, t \in \{1,2,\dots,10\}
\end{aligned}
$$

Facility location

You must locate three bus terminals in a city with a grid network. You know the locations of customers throughout the city, and the cost of building a terminal at each intersection. Passengers walk from their home locations to the nearest terminal. Where should the terminals be located to minimize total walking distance and construction cost?



Let $x(i)$ and $y(i)$ be the coordinates of intersection i (out of I in total). Let H_p be the intersection corresponding to the home location of passenger p, and L_1, L_2, and L_3 the intersections where the three terminals are located. Let $C(i)$ be the cost of building a terminal at i.

The walking distance between intersections i and j is

$$d(i,j) = |x(i) - x(j)| + |y(i) - y(j)|$$

So, the walking distance for customer p is

$$D(p, L_1, L_2, L_3) = \min\{d(H_p, L_1), d(H_p, L_2), d(H_p, L_3)\}$$

How do we write the objective? There are two parts, total walking distance $\sum_{p \in P} D(p, L_1, L_2, L_3)$ and construction cost $C(L_1) + C(L_2) + C(L_3)$.

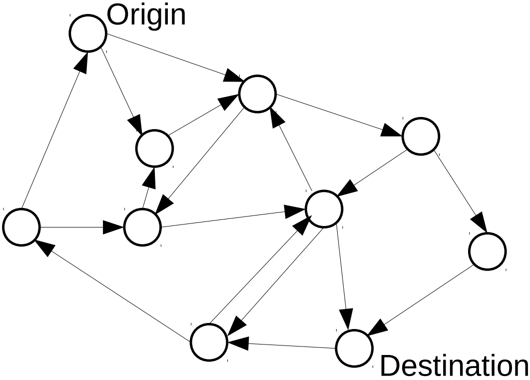
One "trick" is to add them together, using the weighting parameter $\Theta \in [0, 1]$ to show how important each objective is:

$$f(L_1, L_2, L_3) = \Theta[C(L_1) + C(L_2) + C(L_3)] + (1 - \Theta) \left[\sum_{p \in P} D(p, L_1, L_2, L_3) \right]$$

$$
\begin{aligned}
\min_{L_1, L_2, L_3} \quad & \Theta[C(L_1) + C(L_2) + C(L_3)] + (1 - \Theta) \left[\sum_{p \in P} D(p, L_1, L_2, L_3) \right] \\
\text{s.t.} \quad & L_f \in \{1, 2, \ldots, I\} \qquad \forall f \in \{1, 2, 3\}
\end{aligned}
$$

Shortest path

Each roadway link in the network has a known travel time. What is the fastest route connecting the origin to the destination?



Number the roadway links from 1 to A and the intersections from 1 to I. Let $x_a = 1$ if link a is part of the route and 0 otherwise.

The travel time on the route is then $\sum_{a=1}^{A} t_a x_a$.

What are the constraints?

For the path to be valid, we need *flow conservation constraints* at each intersection.

Let $F(i)$ and $R(i)$ be the links leaving and entering intersection i.

For any intersection i, what can we say about how the x values for links in $F(i)$ and $R(i)$ are related?

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \sum_{a \in A} t_a x_a \\
\text{s.t.} \quad & \sum_{a \in F(i)} x_a - \sum_{a \in R(i)} x_a = \begin{cases} 1 & \text{if } i = r \\ -1 & \text{if } i = s \\ 0 & \text{otherwise} \end{cases} & \forall i \in \{1, \dots, I\} \\
& x_a \in \{0, 1\} & \forall a \in \{1, \dots, A\}
\end{aligned}
$$

MORE DEFINITIONS AND USEFUL FACTS

A feasible solution $\mathbf{x}^* \in X$ is a *global minimum* of f if $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all feasible $\mathbf{x}$, and a *global maximum* if $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all feasible $\mathbf{x}$

An *optimal* solution is a global minimum (for a minimization problem) or a global maximum (for a maximization problem).

It is easy to convert back and forth between maximization and minimization problems. If the feasible set is X, the feasible solution $\mathbf{x}^*$ is a global maximum of f iff it is a global minimum of $-f$.

As a result, we do not need to develop different techniques for maximization or minimization problems. **The default convention in this class is to work with minimization problems.**

Other useful facts:

- Constants can be added or subtracted to an objective function without changing the optimal solutions.
- You can multiply an objective function by a nonnegative constant without changing the optimal solutions.

INFORMAL ASSIGNMENT

Identify an optimization problem from your life. Define notation, the objective function, decision variables, and constraints. We'll discuss next class.