**CE 377K: Homework 1**

Due Thursday, February 12

**Problem 1**. You are responsible for allocating bridge maintenance funding for a state, and must develop optimization models to assist with this for the current year. Political realities and a strict budget will require you to develop multiple formulations for the purposes of comparison.

Let $\mathcal{B}$ denote the set of bridges in the state. There is an "economic value" associated with the condition of each bridge: the higher the condition, the higher the value to the state, because bridges in worse condition require more routine maintenance, impose higher costs on drivers who must drive slower or put up with potholes, and carry a higher risk of unforeseen failures requiring emergency maintenance. To represent this, there is a value function $V_b(x_b)$ associated with each bridge $b \in \mathcal{B}$, giving the economic value of this bridge if $x_b$ is spent on maintenance this year. You have a total budget of $X$ to spend statewide. The state is also divided into $n$ districts, and each bridge belongs to one district. Let $\mathcal{D}_i$ denote the set of bridges which belong to district $i$; different districts may contain different numbers of bridges.

In the following, be sure to define any additional notation you introduce. There is more than one possible formulation that achieves the stated goals, so feel free to explain any parts of your formulations which may not be self-evident.

1. **Maximize average benefits:** Formulate an optimization model (objective function, decision variables, and constraints) to maximize the average economic value of bridges across the state.
2. **Equitable allocation:** The previous model may recommend spending much more money in some districts than others, which is politically infeasible. Formulate an optimization model based on maximizing the total economic value to the state, but ensuring that each district receives a relatively fair share of the total budget $X$.
3. **Equitable benefits:** There is a difference between a fair allocation of money, and a fair allocation of *benefits* (say, if the functions $V_b$ differ greatly between districts). Defining *benefit* as the difference between the economic value after investing in maintenance, and the "do-nothing" economic value, formulate an optimization model which aims to achieve a relatively fair distribution of benefits among districts.

**Problem 2**. For the following functions, identify all stationary points and global optima.

(a) $f(x) = x^4 - 2x^3$

(b) $f(x_1, x_2) = 2x_1^2 + x_2^2 - x_1x_2 - 7x_2$

(c) $f(x_1, x_2) = x_1^6 + x_2^6 - 3x_1^2x_2^2$

(d) $f(x_1, x_2, x_3) = (x_1 - 4)^2 + (x_2 - 2)^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$

**Problem 3**. Which of the following functions are coercive? Of the first five, which are convex?

(a) $f(x) = x^4 - 2x^3$

(b) $f(x) = 2x^2 - 4x + 3$

(c) $f(x) = -x^2$

(d) $f(x) = 2x$

(e) $f(x) = x + e^x$

(f) $f(x_1, x_2) = 2x_1^2 + x_2^2$

(g) $f(x_1, x_2) = x_1^4 + x_2^4 - 2x_1^2 x_2^2$

**Problem 4**. The golden section method, applicable for finding the minimum of a unimodal function $f$ without needing derivatives.

(a) The bisection method works by using three test points at each iteration $k$: the endpoints $a_k$ and $b_k$, and the midpoint $c_k = (a_k + b_k)/2$ between them. If $f'(c_k)$ exists, we can use it to eliminate one half of the interval or the other. Show that if can only use the value of $f(c_k)$, and not the derivative there, it is not possible to know which half of the interval to eliminate.

(b) The golden section method works by using *four* test points at each iteration $k$, dividing the interval $[a_k, b_k]$ into three subintervals: $[a_k, c_k]$, $[c_k, d_k]$, and $[d_k, b_k]$ where $a_k < c_k < d_k < b_k$. Let the widths of these three intervals be $\alpha_k = c_k - a_k$, $\beta_k = d_k - c_k$, and $\gamma_k = b_k - d_k$. Show that if we know the values $f(a_k)$, $f(b_k)$, $f(c_k)$, and $f(d_k)$, we can safely eliminate either $[a_k, c_k]$ or $[d_k, b_k]$ as not containing the minimum, and use the remainder as the interval $[a_{k+1}, b_{k+1}]$ for the next iteration.

(c) In particular, the golden section algorithm always chooses $c_k$ and $d_k$ so that $\alpha = \gamma = (3-\sqrt{5})(b_k - a_k)/2$ and $\beta = (\sqrt{5} - 2)(b_k - a_k)$. Show that, regardless of whether the upper or lower interval is eliminated, the ratio $(b_{k+1} - a_{k+1})/(b_k - a_k) = (\sqrt{5} - 1)/2$. (This is the *golden ratio*, from which the method gets its name.0

(d) As a very useful byproduct of this choice of $c_k$ and $d_k$, we only have to generate one new point (and one new objective function value) at each subsequent iteration; three of the four points needed can be reused from prior iterations. If the upper interval is eliminated in iteration $k$, show that $d_{k+1} = c_k$, and if the lower level is eliminated show that $c_{k+1} = d_k$.

**Problem 5**. In the programming language of your choice, find the global minima of the following functions using both Newton's method and the bisection method. Run each method for five iterations, and see which is closer to the optimum, making the comparison based on the value of the objective function at the final points.

(a) $f(x) = -\arctan x$, $x \in [0, 10]$

(b) $f(x) = x \sin(1/(100x))$, $x \in [0.015, 0.04]$

(c) $f(x) = x^3$, $x \in [5, 15]$