CE 377K: Homework 1
Solutions

Problem 1. Many potential answers, but here is one possibility.

1. Decision variables: x_b, objective is to maximize $\sum_{b \in \mathcal{B}} V_b(x_b)$, constraints are $x_b \geq 0$ for all b and $\sum_{b \in \mathcal{B}} x_b \leq X$.
2. The objective function and decision variables are the same as before, and we will retain the same constraints and add a few more. We can represent equitable allocation by dividing the budget based on the number of bridges in each district (denoted $|\mathcal{D}_i|$). We will forbid any district from using more than 110% of its fair share of the budget, represented by adding constraints of the form $\sum_{b \in \mathcal{D}_i} x_b \leq 1.1X \left(|\mathcal{D}_i| / \sum_{j=1}^n |\mathcal{D}_j| \right)$ for each district $i \in \{1, \cdots, n\}$.
3. Start with the same optimization problem as in the first part, and add constraints restricting the average benefit to each district to be no more than 110

Problem 2.

(a) $f'(x) = 4x^3 - 6x^2$, which vanishes when $x = 0$ or $x = 3/2$. Since f is coercive the global minimum occurs at a stationary point, and since $f(3/2) = -1.69 < 0 = f(0)$, the global optimum is $x = 3/2$.

(b) $\nabla f(x) = [4x_1 - x_2, -x_1 + 2x_2 - 7]^T$, which is equal to the zero vector if $x_1 = 1$ and $x_2 = 4$. The function is coercive so this must be the global optimum.

(c) $\nabla f(x) = [6x_1^5 - 6x_1x_2^2, -6x_1^2x_2 + 6x_2^5]^T$, which is equal to the zero vector if $x_1 = x_2 = 0$, or for any combination of $x_1 = \pm 1$ and $x_2 = \pm 1$. Testing each of these five values, there are four global minima: $(\pm 1, \pm 1)$.

(d) $\nabla f(x) = [2(x_1 - 4) + x_2 + x_3, 2(x_2 - 2) + x_1 + x_3, 2x_3 + x_1 + x_2]^T$, which vanishes at $(5/2, 1/2, -3/2)$. Since f is coercive this must be the global optimum.

Problem 3.

(a) Coercive, since x^4 dominates as $x \to \pm\infty$, but not convex since $f''(x) = 12x^2 - 12x$ is negative if $x = 1/2$.

(b) Coercive, since x^2 dominates as $x \to \pm\infty$, and convex since $f''(x) = 4 > 0$ for all x.

(c) Neither coercive (since $\lim_{x \to \infty} f(x) = -\infty$) nor convex ($f''(x) = -2 < 0$ everywhere).

(d) Not coercive ($\lim_{x \to -\infty} f(x) = -\infty$), but convex ($f''(x) = 0$)

(e) Not coercive ($\lim_{x \to -\infty} f(x) = \infty$), but convex ($f''(x) = e^x > 0$)

(f) Coercive, since $2x_1^2 + x_2^2 \geq x_1^2 + x_2^2 = |\mathbf{x}|$ so surely $f(\mathbf{x})$ tends to infinity if $|\mathbf{x}|$ does.

(g) Not coercive, since $f(x_1, x_2) = 0$ if $x_1 = x_2$. So if these go to infinity together f does not tend to infinity.

Problem 4.

(a) See figures, in the two cases the function values at the endpoints and midpoint are the same, but the optimum occurs on different sides of the midpoint.

(b) There are three possibilities: $f(c_k) > f(d_k)$, $f(c_k) < f(d_k)$, and $f(c_k) = f(d_k)$. In the first case, since f is unimodal we must also have $f(a_k) \geq f(c_k)$, in which case the lower interval $[a_k, c_k)$ can be eliminated. In the second case, again by unimodality we have $f(b_k) \geq f(d_k)$, so the upper interval $(d_k, b_k]$ can be eliminated. In the third case, the global minimum must lie between c_k, and d_k, so either the upper or lower interval can be safely removed.

(c) Because $\alpha = \gamma$, regardless of whether the upper or lower interval is eliminated, the ratio of the new interval width to the old is $(alpha + \beta)/(\alpha + \beta + \gamma)$. Substituting the given values for α, β, and γ provides the answer. If the lower interval is eliminated, then $a_{k+1} = c_k$, so

$$\begin{aligned} c_{k+1} &= a_{k+1} + \frac{3-\sqrt{5}}{2}(b_{k+1} - a_{k+1}) && (1) \\ &= c_k + \frac{3-\sqrt{5}}{2}\frac{\sqrt{5}-1}{2}(b_k - a_k) && (2) \\ &= c_k + (\sqrt{5}-2)(b_k - a_k) && (3) \\ &= d_k && (4) \end{aligned}$$

A parallel derivation provides the result when the upper interval is eliminated.

Problem 5. See attached code.