**CE 377K: Homework 3**
Solutions

**Problem 1**. Using the augmenting path algorithm, we first identify path 1-3-4-5-7-8, which has a capacity two. In the resulting residual graph, the path 1-2-6-8 exists, also with capacity two. At this point we are shipping four units of flow from 1 to 8; since arcs (7,8) and (6,8) form a cut of capacity 4, this is maximal. Increasing capacity on either one of these arcs would allow more flow between 1 and 8.

**Problem 2**. Choose nodes 1 and 2 as the source and sink; the shortest path is [1,2] which has capacity 2, so augment 2 units of flow on this path and update the supply and demand of nodes 1 and 2 to +2 and −2. Again choose nodes 1 and 2 as the source and sink; the shortest path in the residual network is [1,3,4,2] which has capacity 3, so augment 2 units of flow on this path and update the supply and demand of nodes 1 and 2 to zero. Choose nodes 6 and 8 as the source and sink; the shortest path in the residual network is [6,7,8] with capacity 2, so augment 2 units of flow and update the supply and demand of nodes 6 and 8 to +1 and 0. Choose nodes 6 and 7 as the source and sink; the shortest path in the residual network is [6,7] with capacity 1, so augment 1 unit of flow and update the supply and demand of nodes 6 and 7 to 0 and −1. Finally choose nodes 3 and 7 as the source and sink. **The shortest path in the residual graph is [3,4,6,8,7] — note the presence of a reverse link with negative cost.** We can send one unit of flow along this link. This provides the flow shown in Figure 1. All sources and sinks are balanced, so this is the optimal solution.

**Problem 3**. Label the equations as follows:

$$x_1 + x_2 + x_3 = 1 \tag{1}$$
$$3x_1 + 2x_2 - x_4 = 3 \tag{2}$$
$$2x_1 + x_3 + x_4 \le 5 \tag{3}$$
$$2x_2 + x_3 + x_4 \le 7 \tag{4}$$
$$x_1 \ge 0 \tag{5}$$
$$x_2 \ge 0 \tag{6}$$

1. Constraints (1), (2), (5), and (6) are active at the extreme point $(0, 0, 1, -3)$ (1) and (2) must be active at any feasible solution, beacuse they are equality constraints. Thus, the only potential extreme points adjacent to $(0, 0, 1, -3)$ lie at the intersection of constraints (1), (2), (3), and (6); constraints (1), (2), (4), and (6); constraints (1), (2), (3), and (5); and constraints (1), (2), (4), and (5). Solving these systems of equations simultaneously, we obtain the points $(1.75, 0, -0.75, 2.25)$, $(4.5, 0, -3.5, 10.5)$, $(0, 7, -6, 11)$, and $(0, 3, -2, 3)$, respectively. $(4.5, 0, -3.5, 10.5)$ and $(0, 7, -6, 11)$ are infeasible; the remaining points are all feasible, and thus extreme points adjacent to $(0, 0, 1, -3)$.
2. The only other potential combination of active constraints is (1), (2), (3), and (4); solving these simultaneously gives the feasible point $(1.2, 2.2, -2.4, 5)$, which is also an extreme point of $X$.

**Problem 4**.

(a) Let $\tau_i$ be the toll in hour $i$. The optimization problem can be stated

$$\begin{aligned}
\max_{\tau_1,\tau_2,\tau_3,\tau_4} \quad & (2500 - 500\tau_1) + (3100 - 1000\tau_2) + (4000 - 1500\tau_3) + (3400 - 1000\tau_4) \\
\text{s.t.} \quad & 2500 - 500\tau_1 \leq 2000 \\
& 3100 - 1000\tau_2 \leq 2000 \\
& 4000 - 1500\tau_3 \leq 2000 \\
& 3400 - 1000\tau_4 \leq 2000 \\
& |\tau_1 - \tau_2| \leq 0.25 \\
& |\tau_2 - \tau_3| \leq 0.25 \\
& |\tau_3 - \tau_4| \leq 0.25 \\
& \tau_1, \tau_2, \tau_3, \tau_4 \geq 0
\end{aligned}$$

(b) Conversion to standard form requires changing maximize to minimize, replacing the absolute values by two equivalent constraints ($|x - y| \leq z$ is the same as both $x - y \leq z$ and $y - x \leq z$), and adding slack variables $\mathbf{s}$. Furthermore, constants can be removed from the objective function and the first four constraints can be simplified.

$$\begin{aligned}
\min_{\boldsymbol{\tau},\mathbf{s}} \quad & 500\tau_1 + 1000\tau_2 + 1500\tau_3 + 1000\tau_4 \\
\text{s.t.} \quad & \tau_1 - s_1 = 1 \\
& \tau_2 - s_2 = 1.1 \\
& \tau_3 - s_3 = 1.33 \\
& \tau_4 - s_4 = 1.4 \\
& \tau_1 - \tau_2 + s_5 = 0.25 \\
& -\tau_1 + \tau_2 + s_6 = 0.25 \\
& \tau_2 - \tau_3 + s_7 = 0.25 \\
& -\tau_2 + \tau_3 + s_8 = 0.25 \\
& \tau_3 - \tau_4 + s_9 = 0.25 \\
& -\tau_3 + \tau_4 + s_{10} = 0.25 \\
& \tau_1, \ldots, \tau_4, s_1, \ldots, s_{10} \geq 0
\end{aligned}$$

(c) See attached code (`homework3.c` has the C code, `tableau.txt` shows the tableaux at each step). The big $M$ method was used, introducing four auxiliary variables so that the initial basis was these four variables, and $s_5, \ldots, s_10$. The optimal solution is $\tau_1 = 1$, $\tau_2 = 1.1$, $\tau_3 = 1.33$, and $\tau_4 = 1.4$.

(d) Refer to the final tableau in `tableau.txt`. The optimal basis for this problem includes all four toll variables (indicating each should be set at their lowest possible level), and slack variables for each of the constraints limiting the change in tolls between periods (indicating that none of these constraints are currently binding). Notice that we turned each of the capacity constraints into $\tau_i \geq K_i \equiv (D_i - C_i)/S_i$ where $D_i$ is the total demand in period $i$, $C_i$ is the capacity in period $i$, and $S_i$ is the toll sensitivity in period $i$. The sensitivity analysis methods from class will tell us the maximum allowable change in $K_i$ before the basis changes, and we will use the formula $K_i = (D_i - C_i)/S_i$ to calculate the corresponding changes in capacity since $C_i = D_i - K_i S_i$, so $\Delta C_i = -S_i \Delta K_i$.

For example, in the first time period, the "original" constraint $2500 - 500\tau_1 \leq 2000$ has been transformed into $\tau_1 \geq 1$. For the sensitivity analysis to changing the right-hand side of this first constraint, look at the 15th column of the tableau (because this is the first column of the identity matrix, correpsonding to

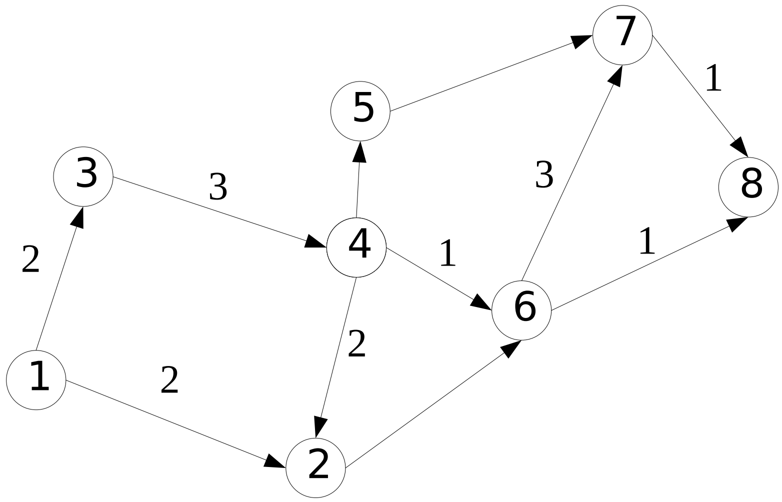


Figure 1: Solution for problem 2.

the first auxiliary variable for the big $M$ method.) Comparing the ratios of each basic decision variables to the tableau entry in the 15th column, $K_1$ can increase by up to 0.35 or decrease by up to 0.15 before the basis changes; this corresponds to capacity changes of $\Delta C_i = 500\Delta K_i$ in the range $[-175, 75]$. These bounds correspond to $s_6$ and $s_5$, respectively; the limit that the toll cannot increase or decrease by more than 25 cents between hours becomes binding if the change in capacity falls outside this range.

Proceeding in a similar way, the allowable ranges for periods 2, 3, and 4 are $[-150, 17]$, $[-25.5, 274.5]$, and $[183, 317]$, respectively.