CE 377K: Homework 4
Due Thursday, May 7

Problem 1. Determine whether the following functions are convex or not. Prove your answer.

1. No. Let $x_1 = 1$, $x_2 = e^2$, $\lambda = (e-1)/(e^2-1)$. Then $f(x_1) = 0$, $f(x_2) = 2$, and $\log((1-\lambda)x_1 + \lambda x_2) = \log e = 1$, which is clearly greater than $(1-\lambda)f(x_1) + \lambda f(x_2) = 2(e-1)/(e^2-1) = 2(e-1)/[(e+1)(e-1)] = 2/(e+1)$.
2. Yes. $d^2 f/dx^2 = -6x \geq 0$ for $x \leq 0$.
3. No. Let $x_1 = -\pi$, $x_2 = 0$, $\lambda = 1/2$. Then $f(x_1) = f(x_2) = 0 \Rightarrow (1-\lambda)f(x_1) + \lambda f(x_2) = 0$, which is greater than $f(-\pi/2) = -1$.
4. Yes. We know e^x, x^2, and $-2x-3$ are convex (check second derivatives). Since the sum of convex function is convex, $x^2 - 2x - 3$ is as well. Since the composition of convex function is convex, e^{x^2-2x-3} is convex as well.

Problem 2. Determine whether the following sets are convex or not. Prove your answer.

1. Yes. This is a hyperplane in the form of Example 10 (with $a_1 = 4$, $a_2 = -3$, $b = 0$), and thus is convex.
2. Yes. This set is the same as $X = \{x \in \mathbb{R} : e^x - 4 \leq 0\}$. Since $e^x - 4$ is a convex function, X is a convex set.
3. Yes. Choose any $x^1 = (0, x_2^1, x_3^1)$, $x^2 = (0, x_2^2, x_3^2) \in X$ and any $\lambda \in [0,1]$. $(1-\lambda)x^1 + \lambda x^2 = (0, (1-\lambda)x_2^1 + \lambda x_2^2, (1-\lambda)x_3^1 + \lambda x_3^2)$ which is in X because its first component is zero.
4. No. Let $x^1 = (-1,0,0)$, $x^2 = (1,0,0)$, $\lambda = 1/2$. Then $(1-\lambda)x^1 + \lambda x^2 = (0,0,0) \neq X$.

Problem 3. You've been called upon to install a traffic signal at 46th & Guadalupe. Assuming a simple two-phase cycle (where Guadalupe moves in phase 1, and 46th Street in phase 2), no lost time when the signal changes, and ignoring turning movements, the total delay at the intersection can be written as

$$\frac{\lambda_1(c-g_1)^2}{2c\left(1-\frac{\lambda_1}{\mu_1}\right)} + \frac{\lambda_2(c-g_2)^2}{2c\left(1-\frac{\lambda_2}{\mu_2}\right)}$$

where g_1 and g_2 are the effective green time allotted to Guadalupe and 46th Street, respectively; $c = g_1 + g_2$ is the cycle length, λ_1 and μ_1 are the arrival rate and saturation flow for Guadalupe, and λ_2 and μ_2 are the arrival rate and saturation flow for 46th Street.

The cycle length must be 60 seconds to maintain progression with adjacent signals, and the arrival rate and saturation flow are 2200 veh/hr and 3600 veh/hr for Guadalupe, and 300 veh/hr and 1900 veh/hr for 46th Street. Furthermore, no queues can remain at the end of the green interval; this means that $\mu_i g_i$ must be at least as large as $\lambda_i c$ for each approach i.

1. Using the notation in the problem, we seek a solution to

$$\begin{aligned}
\min_{g_1, g_2} \quad & \sum_{i=1}^{2} \frac{\lambda_i (c - g_i)^2}{2c\left(1 - \frac{\lambda_i}{\mu_i}\right)} \\
\text{s.t.} \quad & \sum_{i=1}^{2} g_i = c \\
& \mu_i g_i \geq \lambda_i c & \forall i \in \{1, 2\} \\
& g_i \geq 0 & \forall i \in \{1, 2\}
\end{aligned}$$

   Substituting specific numerical values and eliminating the redundant nonnegativity constraints, we have:

$$\begin{aligned}
\min_{g_1, g_2} \quad & 47.14(60 - g_1)^2 + 2.969(60 - g_2)^2 \\
\text{s.t.} \quad & g_1 + g_2 = 60 \\
& g_1 \geq 36.7 \\
& g_2 \geq 9.47
\end{aligned}$$

2. At the initial solution $g_1 = 45$, $g_2 = 15$, the value of the objective function is 16618, and the gradient is $[-1414, -267]$. So, the linearized version of the objective function is $16618 - 1414(g_1^* - 45) - 267(g_2^* - 15)$, to be minimized over the same constraints as in the original problem. Removing constants, we need to minimize $-1414g_1^* - 267g_2^*$ subject to $g_1 + g_2 = 60$, $g_1 \geq 36.7$, and $g_2 \geq 9.47$. This is a linear program with only two extreme points — (36.7, 23.3) and (51.53, 9.47) — so one of these must be an optimal solution. Testing both shows that the latter point is optimal, so the target solution is (51.53, 9.47). With the method of successive averages, we take a step of size $\lambda = 1/2$ towards this target, giving the solution $g_1 = 48.26$ and $g_2 = 12.74$.

   At this new solution, the value of the objective function is 13407, and the gradient is $[-1106, -287]$. After removing constants, the new linearized objective function is to minimize $-1106g_1^* - 267g_2^*$. Checking the two extreme points, the target solution is again (51.53, 9.47). Taking a step of size $\lambda = 1/3$ gives a solution $g_1 = 49.35$, $g_2 = 10.65$, which has an objective function value of 12577.

3. Introducing a Lagrange multiplier κ for the equality constraint and two KKT multipliers π_1 and π_2 for the minimum cycle length constraints, the KKT conditions for this problem are as follows:

$$\begin{aligned}
94.29g_1 + \kappa - \pi_1 &= 5657 && (1) \\
5.938g_2 + \kappa - \pi_2 &= 356.2 && (2) \\
g_1 + g_2 &= 60 && (3) \\
g_1 &\geq 36.7 && (4) \\
g_2 &\geq 9.47 && (5) \\
\pi_1, \pi_2 &\geq 0 && (6)
\end{aligned}$$

   with $\pi_1 = 0$ unless $g_1 = 36.7$ and $\pi_2 = 0$ unless $g_2 = 9.47$. So, there are three cases:

(a) Neither of the inequality constraints is binding; in this case $\pi_1 = \pi_2 = 0$. Solving equations 1–3 simultaneously yields $g_1 = 56.44$, $g_2 = 3.58$, $\kappa = 335$. This violates constraint 5, so this cannot be an optimal solution.

(b) The constraint $g_1 \geq 36.7$ is binding. This means $g_1 = 36.7$, so $g_2 = 23.3$ by equation 3, and therefore $\pi_2 = 0$. Substituting these into equation 2 gives $\kappa = 217.8$. Substituting this and $g_1 = 36.7$ into equation 1 gives $\pi_1 = -1978$ which violates the requirement that KKT multipliers on inequality constraints be nonnegative, so this cannot be an optimal solution either.

(c) The constraint $g_2 \geq 9.47$ is binding. This means $g_2 = 9.47$, so $g_1 = 50.53$ by equation 3, and therefore $\pi_1 = 0$. Substituting these into equation 1 gives $\kappa = 892.5$. Substituting this and $g_2 = 9.47$ into equation 2 gives $\pi_2 = 592.6$. This solution satisfies all of the constraints, so it is the optimal solution.

The optimal solution to the problem is $g_1 = 51.53$ and $g_2 = 9.47$, which has an objective function value of 11807.