Higher-order continuum flow models

CE 391F

February 21, 2013

ANNOUNCEMENTS

- Homework 2 coming
- Paper presentations in two weeks

Paper presentations

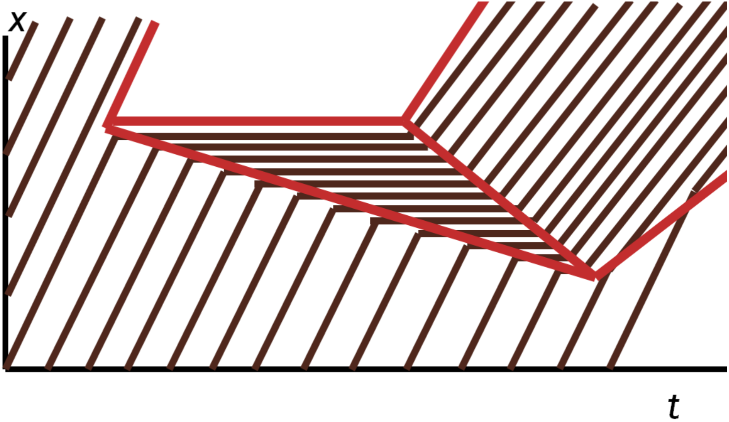
- Each student picks one paper from the list on the course website
- Prepare a 15-minute presentation based on that paper, teaching the main ideas to the class
- Grading will be based on technical correctness, skill at conveying information, and delivery.

REVIEW

- “First-order” LWR model
- Shockwaves and spreading

TOWARDS SECOND-ORDER MODELS

Why don't shocks occur instantaneously?



Drivers have reaction time, and may also anticipate downstream conditions. Let's look at these one at a time.

Diffusion: awareness of downstream conditions

To date, the density at a point has been the sole determinant of the flow $Q(k)$ and speed $u(k)$. We want to add "look-ahead" behavior.

One way to do this is to say that the flow at a point is given by

$$q = Q(k) - c_0^2 k_x$$

where c_0^2 is a constant and $k_x = \partial k / \partial x$ gives the (spatial) rate at which density is changing.

Why c_0^2? As we will see, we can draw analogies with fluid mechanics and the wave equation, and this is a standard notation.

$$q = Q(k) - c_0^2 k_x$$

What does this formula predict when $k_x > 0$? Is this logical?

What about when $k_x < 0$?

$k_x = 0$?

Dividing by k, we obtain a relationship between speed and density:

$$u = U(k) - c_0^2 \frac{k_x}{k}$$

where $U(k) = Q(k)/k$ is the speed-density relationship based on the fundamental diagram.

In both this formula and the previous one, $Q(k)$ and $U(k)$ give the "steady-state" flows and speeds associated with the density k. The actual flow and speeds q and u have an additional term to reflect drivers looking downstream and anticipating future conditions.

To introduce the effect of reaction time, we introduce a time lag into the speed-density formula:

$$u(x, t + \tau) = U(k(x, t)) - c_0^2 \frac{k_x(x, t)}{k(x, t)}$$

where τ is the reaction time.

Taking a linear approximation to the left-hand side, we have

$$u(x, t + \tau) \approx u(x, t) + \tau \frac{du}{dt} = U(k(x, t)) - c_0^2 \frac{k_x(x, t)}{k(x, t)}$$

or

$$\frac{du}{dt} = \frac{1}{\tau} \left(U(k) - u - \frac{k_x}{k} \right)$$

omitting the x and t indices for clarity.

$$\frac{du}{dt} = \frac{1}{\tau}\left(U(k) - u - c_0^2 \frac{k_x}{k}\right)$$

In this formula, you can interpret $[U(k) - k_x/k] - u$ as the difference between the "desired" and "actual" speed at a point.

As τ decreases, drivers react faster (du/dt grows large.)

As τ increases, drivers react slower. What happens as $\tau \to 0$ or $\tau \to \infty$?

Keep in mind that this interpretation for du/dt is in terms of a particular vehicle (which is *moving*)

How does this relate to the derivatives of $u(x,t)$ at a single point in time and space?

For a moving vehicle x is a function of t, so we actually have

$$\frac{du(x(t),t)}{dt} = \frac{1}{\tau}\left(U(k) - u - c_0^2\frac{k_x}{k}\right)$$

Applying the chain rule, the left-hand side is

$$\frac{du}{dx}\frac{dx}{dt} + \frac{du}{dt}$$

or $u_x u + u_t$

This gives the *momentum equation*

$$u_x u + u_t = \frac{1}{\tau}\left(U(k) - u - c_0^2 \frac{k_x}{k}\right)$$

The *conservation equation* still holds true:

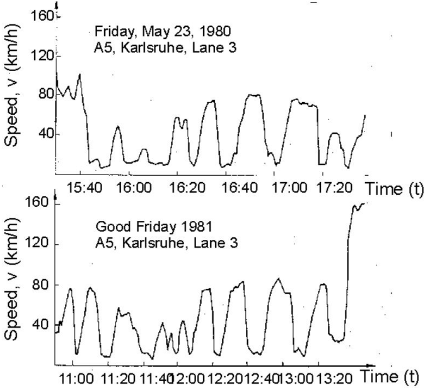
$$k_t + q_x = 0$$

The basic LWR model is “first-order” because it only has the conservation equation. “Second-order” models include the momentum equation as well.

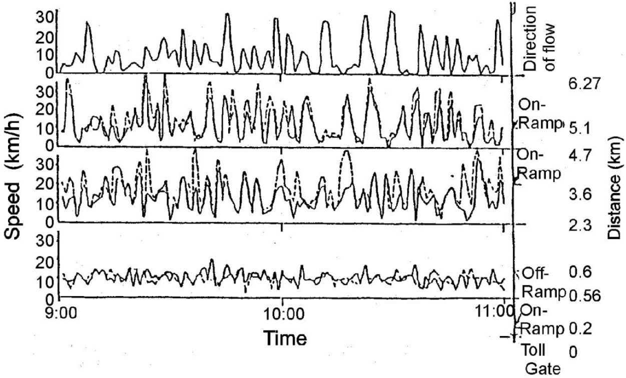
This is called the PW (Payne-Whitham) model, and was developed in the early 1970s.

EMPIRICAL SUPPORT

In addition to smoothing shocks, higher-order models can reproduce other phenomena observed in the field:

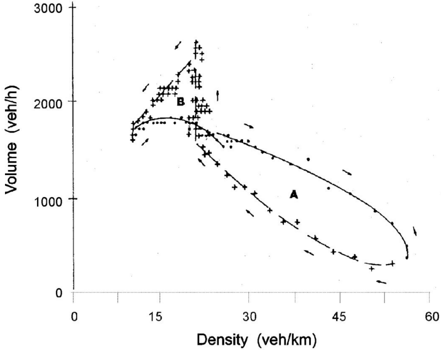


Congested traffic exhibits oscillation in speed, rather than smooth flow ("stop-and-go")



30
20
10
0
Speed (km/h)
9:00
10:00
11:00
Time
Direction of flow
On-Ramp
Off-Ramp
Toll Gate
6.27
5.1
4.7
3.6
2.3
0.6
0.56
0.2
0
Distance (km)

Traffic congestion also exhibits *hysteresis*: the speed-density diagram seems to be different when flow is recovering from congestion and when congestion is forming.



This is impossible to recreate with a first-order model, but can (somewhat) be modeled using the $c_0^2 k_x$ term.

RELATIONSHIP WITH FLUID DYNAMICS

The conservation and momentum equations can easily be cast in terms of fluid dynamics or kinetic gas theory

$$u_x u + u_t = \frac{1}{\tau}\left(U(k) - u - c_0^2 \frac{k_x}{k}\right)$$

$$k_t + q_x = 0$$

(Perhaps too easily.)

We'll survey some of the ways this has been done.

In kinetic gas theory, atoms are in constant motion and collide with each other elastically.

Using this theory, one can in fact derive the momentum equation with the interpretation of u as the average speed and c_0^2 as the variance.

Furthermore, by combining the conservation and momentum equations, one can derive the wave equation

$$u_{tt} - c_0^2 u_{xx} = 0$$

where c_0 can represent the “speed of sound” (the rate at which disturbances propagate)

The momentum equation can also be cast as a special case of the Navier-Stokes equations

$$u_x u + u_t = \frac{1}{\tau}\left(U(k) - u - c_0^2 \frac{k_x}{k}\right)$$

in which $-c_0^2 k_x / k$ is interpreted as the "traffic pressure."

This "traffic pressure" manifests itself as drivers anticipate changes ahead of them.

Can you see any difficulty with this analogy?

The momentum equation can also be augmented with a *viscosity term*

$$u_x u + u_t = \frac{1}{\tau}\left(U(k) - u - c_0^2 \frac{k_x}{k}\right) + \frac{1}{k}\left(\mu_0 u_x\right)_x$$

where μ_0 represents the dynamic viscosity associated with shear forces.

Then we can introduce the Reynolds number $\mathbb{R} = u_f^2 \tau^2 / \mu_0$ and Froude number $\mathbb{F} = (u_f / c_0)^2$

A SOURCE OF CONTROVERSY

These parallels have proven nearly irresistable, but some researchers remain unconvinced that these lines of investigation are worthwhile.

Why?

- While conservation holds for vehicles, “momentum” and “viscosity” do not.
- Vehicles (hopefully) are not involved in frequent, elastic collisions.
- Higher-order models can violate the *anisotropic principle*.
- They are considerably harder to calibrate and solve.

Daganzo's Requiem

In 1995, Daganzo published "Requiem for second-order fluid approximations of traffic flow" in *Transportation Research Part B*.

He produces a simple example in which vehicles actually have to flow *backwards* to satisfy PW.

Subsequently, traffic flow researchers have built better second-order models to address these criticisms; the debate rages to this day.

INFORMAL EVALUATIONS

1. How is the pace of the class so far? (SLOW / OK / FAST)
2. What topic is most unclear at this point?
3. What about my teaching is most helpful to you?
4. What can I do better?
5. Any other comments?