# Solving higher-order models

CE 391F

February 26, 2013

# ANNOUNCEMENTS

- Homework 2 online
- Paper presentations next week

# REVIEW

# Second-order models

- Why?
- $u_x u + u_t = \frac{1}{\tau}\left(U(k) - u - c_0^2 \frac{k_x}{k}\right)$
- Extensive connections with kinetic gas theory and fluid dynamics
- Somewhat controversial

# SOLVING HIGHER-ORDER MODELS

Today, we'll talk about how to *solve* higher-order models. Recall that we want to solve both the conservation and momentum equations simultaneously

$$k_t + q_x = 0$$

$$u_x u + u_t = \frac{1}{\tau}\left(U(k) - u - c_0^2 \frac{k_x}{k}\right)$$

given appropriate boundary conditions.

As we will see, this is considerably more difficult than the first-order models, for several reasons.

What makes this problem harder?

- It's a system of PDEs, not just a single one
- Can't eliminate all variables except $k$ because density doesn't map directly to flow or speed.
- Practical experience shows that extremely small $\Delta x$ and $\Delta t$ values are needed for stability using the solution methods we've seen so far.

A simple discretization and forward simulation (like the cell transmission model) is an *explicit* solution method. When higher accuracy is needed, we often turn to *implicit* solution methods

# EXPLICIT AND IMPLICIT METHODS

Explicit and implicit are a classification of numerical methods for differential equations.

As a first example, consider the simple differential equation $\frac{dy}{dx} = y$, where $y(0) = 1$ and we want to solve for positive values of $x$.

(What's the solution to this differential equation?)

Both explicit and implicit methods approximately solve differential equations by calculating the value of $y$ at selected points (say, on $\mathbb{Z}^+$), starting from the boundary, and approximating the derivative $dy/dx$ by finite differences.

In this example, we might try to find $y(1)$, $y(2)$, $y(3)$, and so forth, in that order.

"Explicit" methods have a direct formula for $y(i)$ in terms of $y(i-1)$.

For example, if we know the value of $y(i-1)$, then $y(i) \approx y(i-1) + y'(i-1)$. This is an *explicit* formula for $y(i)$.

What solution does this give for $y' = y$, $y(0) = 1$? How does this compare to the true solution?

The cell transmission model is essentially an explicit method. Starting with the $k$ (or $N$) values at a boundary, we move forward in strict increasing order, with a direct formula for the number of vehicles in each cell based on the previous time step.

The cell size in CTM satisfies the CFL condition, and so it produces reasonably accurate results.

However, for higher-order models, satisfying this condition often requires impractically small values of $\Delta x$ and $\Delta t$

We can make the finite difference approximation better if we calculate the derivative at the *midpoint* of $i$ and $i-1$, rather than just at $i-1$.

We don't know the value of $y(i)$ yet, but whatever it is, we will take $y(i)-y(i-1)$ as an approximation for the derivative at $y(i-1/2)$, and assume that $y$ is roughly linear between $i-1$ and $i$.

Thus, we need to solve the equation $y(i)-y(i-1)=y'(i-1/2)$

For our example, this becomes
$y(i)-y(i-1)=y'(i-1/2)=y(i-1/2)=(y(i)+y(i-1))/2$

Thus, we have an *implicit* equation for $y(i)$ that we need to solve:
$y(i)-y(i-1)=(y(i)+y(i-1))/2$

(Remember, we assume $y(i-1)$ is known and $y(i)$ is unknown.)

For our case, we can solve this equation to get $y(i) = 3y(i-1)$ or $y \approx 3^x$, which is somewhat more accurate than the explicit method that gave $y \approx 2^x$.

Sometimes, it can be more work to solve the implicit equation: what would we get if $y' = y^2$ instead?

We would have to solve the quadratic equation

$$y(i) - y(i-1) = \left( \frac{y(i) + y(i-1)}{2} \right)^2$$

For more difficult equations, it is easier to produce a numerical solution using Newton's method.

1. Rewrite the implicit equation as $F(y(i)) = 0$
2. Provide an initial guess for $y(i)$
3. Update $y(i) \leftarrow y(i) - F(y(i))/F'(y(i))$
4. Repeat last step until $F(y(i)) \approx 0$

Newton's method isn't guaranteed to work, but in practice it often does quite well. Empirically, it works very well for the PW model.

So, the overall implicit method works as follows:

1. Start at the boundary where $y$ is known
2. Determine the next point $y(i)$ using Newton's method (or directly solving the equation), based on the value of the derivative at the *midpoint*, and the values of $y$ already calculated.
3. Move to the next point and continue.

# Towards solving PW

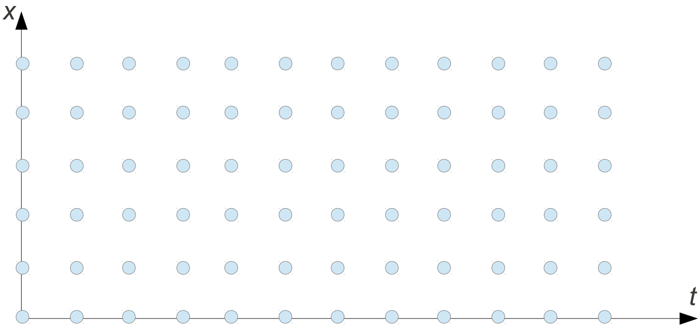
To work with stepsizes other than $\Delta x = 1$, we modify the finite difference approximation $f'(x) \approx (f(x + \Delta x) - f(x))/\Delta x$

In two dimensions, we construct a "lattice" of $(x, t)$ points rather than a one-dimensional line.

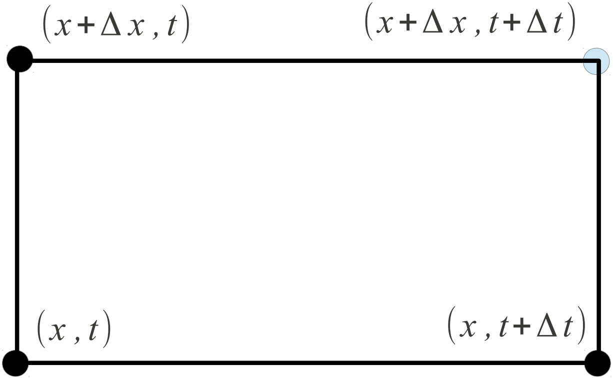
Further, in two dimensions, we need to use the multidimensional Newton's method: if $F : \mathbb{R}^{\ltimes} \to \mathbb{R}^n$ then the update is $y \leftarrow y - F(y)JF^{-1}(y)$ where $JF$ is the Jacobian of $F$ at $(y)$, assuming this matrix is invertible there.

# IMPLICIT METHOD FOR PW MODEL

# We will work in a grid in $(x, t)$ space




We will calculate all derivatives based on the *midpoint* of each rectangle.




So, for instance, $\partial k/\partial x$ at the midpoint is approximated by

$$\frac{1}{2\Delta x}\left(k(x+\Delta x, t+\Delta t) - k(x, t+\Delta t) + k(x+\Delta x, t) - k(x, t)\right)$$

and

$$\frac{\partial k}{\partial t} \approx \frac{1}{2\Delta t}\left(k(x+\Delta x, t+\Delta t) - k(x+\Delta x, t) + k(x, t+\Delta t) - k(x, t)\right)$$

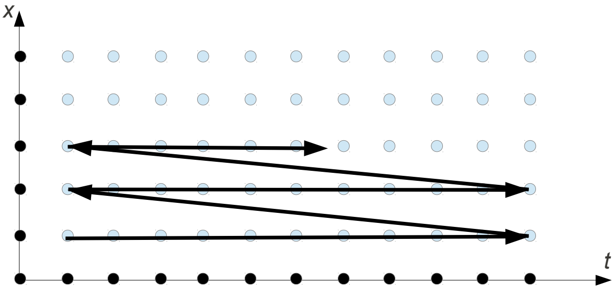
It is customary to reframe the problem in three variables: $k$, $u$, and $w \equiv u_x$

As shorthand, all of these variables are collected in the vector $\eta = \begin{bmatrix} k & u & w \end{bmatrix}$.

Then, we solve the following system of differential equations together:

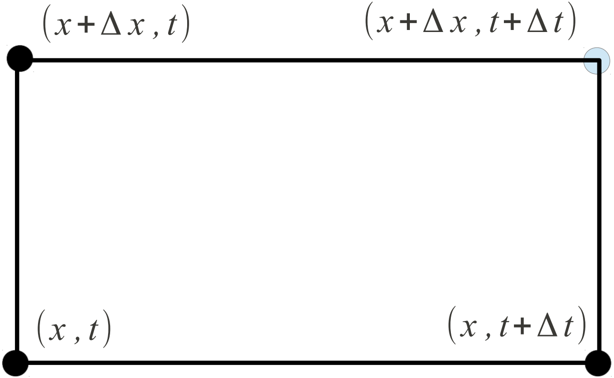
$$\begin{aligned} k_t + uk_x + kw &= 0 \\ u_t + (c_0^2/\tau)k_x/k + uw + (1/\tau)(U(k) - u) &= 0 \\ u_x - w &= 0 \end{aligned}$$

We will solve the problem for the first time interval, moving across space; then for the second time interval, across space, and so forth.




This solution scheme applies when we know the values of $\eta$ ($k$, $u$, and $w$) along the axes $t = 0$ and $x = 0$

With this solution scheme, when we consider each rectangle there is only one unknown point.




Denote the unknown point $\eta^* = \eta(x+\Delta x, t+\Delta t)$.

Thus, the derivatives become

$$\eta_x(\eta^*) = \frac{1}{2\Delta x}\left(\eta^* - \eta(x, t+\Delta t) + \eta(x+\Delta x, t) - \eta(x,t)\right)$$

and

$$\eta_t(\eta^*) = \frac{1}{2\Delta t}\left(\eta^* - \eta(x+\Delta x, t) + \eta(x, t+\Delta t) - \eta(x,t)\right)$$

All values of $\eta$ are calculated based on the "midpoint"
$\overline{\eta}(\eta^*) = \frac{1}{4}(\eta(x,t) + \eta(x+\Delta x, t) + \eta(x, t+\Delta t) + \eta^*)$

So, we need to solve the following system of equations for $\eta^*$, using the multivariate Newton's method:

$$\begin{aligned} k_t(\eta^*) + u(\eta^*)k_x(\eta^*) + k(\eta^*)w(\eta^*) &= 0 \\ u_t(\eta^*) + (c_0^2/\tau)k_x(\eta^*)/k(\eta^*) + u(\eta^*)w(\eta^*) + (1/\tau)(U(k(\eta^*)) - u(\eta^*)) &= 0 \\ u_x(\eta^*) - w(\eta^*) &= 0 \end{aligned}$$

# HOMEWORK ORIENTATION

**Problem 1**. Consider a roadway with a trapezoidal fundamental diagram, with free-flow speed 30 mph, capacity 1800 veh/hr, jam density 240 veh/mi, and backward wave speed 15 mph. The inflow rate has a triangular profile: the inflow rate is $90t$ veh/hr for $0 \leq t \leq 15$ ($t$ measured in minutes), $90(30 - t)$ veh/hr for $15 \leq t \leq 30$, and 0 afterwards. This inflow rate is measured 1 mile upstream of a traffic signal with a 60-second cycle length, starting at $t = 0$ with 15 seconds of effective green followed by 45 seconds of effective red; assume that the fundamental diagram is not affected by turning vehicles. For all parts of this problem, use the cell transmission model with a timestep of 15 seconds.

(a) What is the time at which the last vehicle passes the signal?

(b) Estimate the maximum queue length which will occur.

(c) What is the average travel time for the vehicles on this roadway, measuring travel time from 1 mile upstream of the signal to immediately downstream of the signal?

(d) Now, assume that the signal timing can be changed: the cycle length remains 60 s, but the effective green and red times can be changed (but must be multiples of 15 seconds). What is the minimum effective green time which would be needed to ensure that the queue *on this approach* is always cleared at the start of each red interval?

**Problem 2**. Continuing Problem 1, assume that this signal lies at the intersection of two one-way roads. One was described in Problem 1; the other has an identical fundamental diagram, but a slightly different inflow rate: a constant 1200 veh/hr for $0 \leq t \leq 30$ ($t$ in minutes), again measured 1 mile upstream of the signal. The effective green time for each approach corresponds exactly to the effective red time to the other approach.

(a) Returning to the initial scenario in Problem 1 (the first approach has 15 seconds of green and 45 seconds of red), what is the average travel time if we account for vehicles on *both* roadways (again measuring travel times from 1 mile upstream of the signal to immediately downstream).

(b) Propose a revised signal timing for reducing average travel time. You may change either the cycle length or the allocation of green and red times, as long as all values are multiples of 15 seconds, and the signal turns green for the first approach at $t = 0$. (These values must remain constant over the modeling period.) What is the average travel time for your revised timing? *10 extra credit points will be awarded to the best timing in the class, divided among the number of students who found that value.*

**Problem 3**. As discussed in class, a single loop detector can only measure flow and occupancy (the fraction of time a vehicle overlaps any part of the detector). Making some assumptions about vehicle length, we can relate this to other traffic variables. Assume that all vehicles have the same length $L$, and that the detector itself has a length of $d$.

(a) If a detector measures a vehicle on top of it for $t$ seconds, what speed is the vehicle traveling?

(b) If we observe $n$ vehicles passing the detector in a time interval $T$, find an equation relating the occupancy $O$ to the space-mean speed $\overline{u}_s$ of the vehicles and the flow $q$; then use the fundamental relationship to find an equation relating density to occupancy.