Lighthill-Whitham-Richards Model

CE 391F

January 24, 2013

OUTLINE

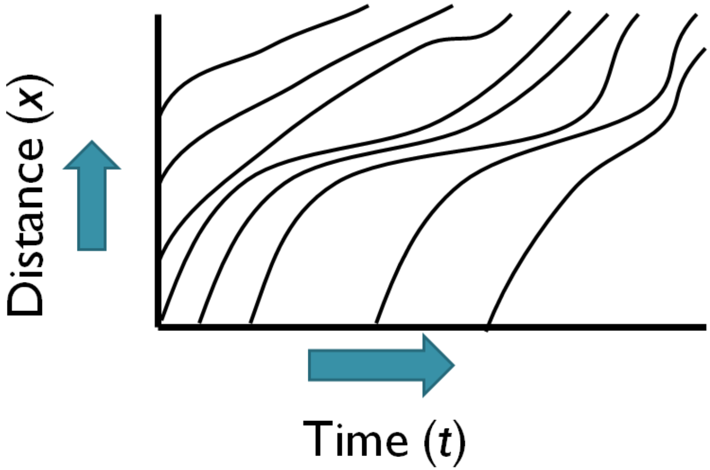
1. Review
2. Fundamental relationship and fundamental diagram
3. PDE formulation of LWR model

REVIEW

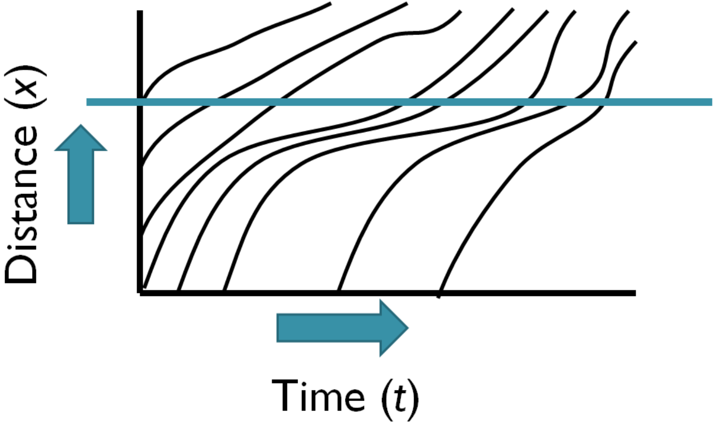
Fundamental quantities...

- q
- k
- u
- $\overline{u}$
- Headways
- N

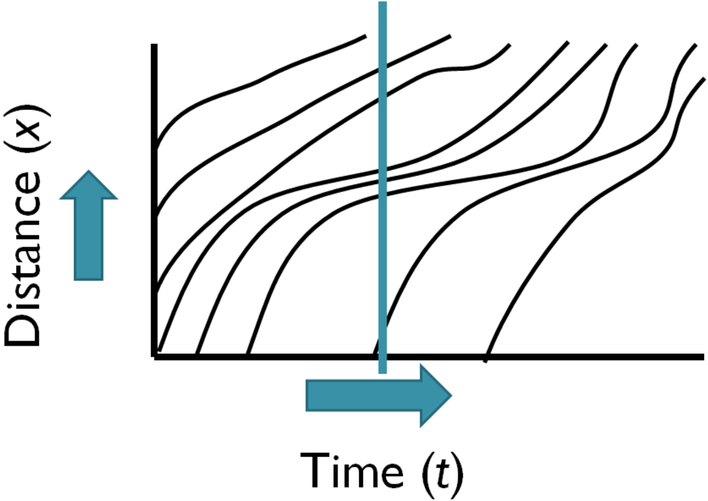
These quantities can be visualized on a *trajectory diagram*.



The flow q is the rate trajectories cross a horizontal line.



The density k is the rate trajectories cross a vertical line.

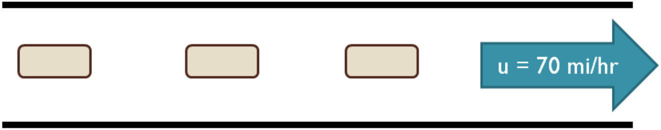


How is speed represented in a trajectory diagram?

RELATING SPEED, VOLUME, AND DENSITY

How are these three quantities related to each other?

I sit at the side of the road for one hour, while cars drive by at 70 mi/hr. If the density is 10 veh/mi, how many vehicles pass by?



k = 10 veh/mi

In one hour, every car within 70 mi will drive by. Since the density is 10 veh/mi, I will see 700 vehicles.

This is a demonstration of the *fundamental relationship* between speed, flow, and density:

$$q = uk$$

Think units: [veh/hr] = [mi/hr][veh/mi]

How are flow and density related?

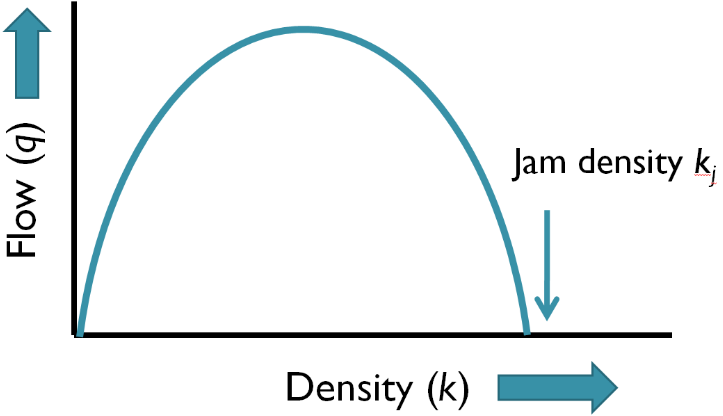
If $q = 0$, *either* $k = 0$ or $u = 0$.

If $u = 0$, then the density is at its maximum value k_j, the *jam density*.

So, $q = 0$ if $k = 0$ or $k = k_j$.

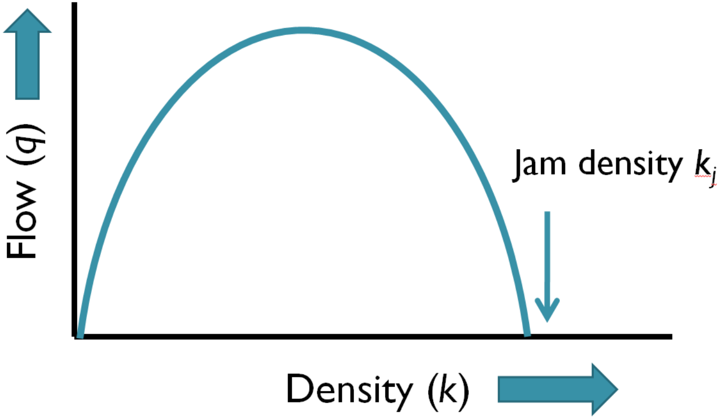
The major assumption in the LWR model is that q is a function of k alone. (k determines u, so $q(k) = u(k) \cdot k$)

Therefore, $Q(k)$ must be a concave function with zeros at $k = 0$ and $k = k_j$



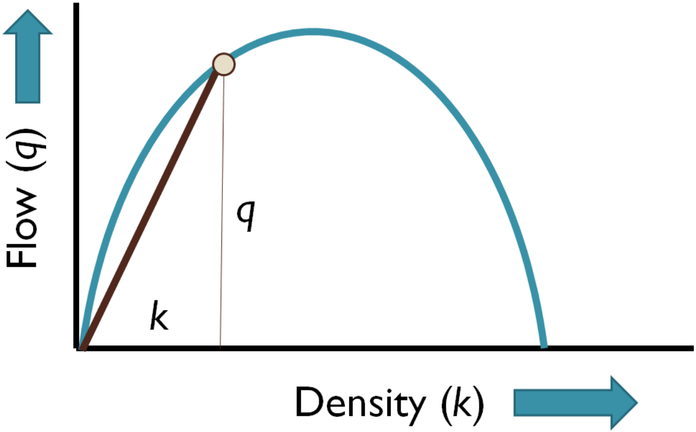
This is the *fundamental diagram*, denoted $Q(k)$. The fundamental diagram can be calibrated to data, resulting in different traffic flow models.

Therefore, $Q(k)$ must be a concave function with zeros at $k = 0$ and $k = k_j$

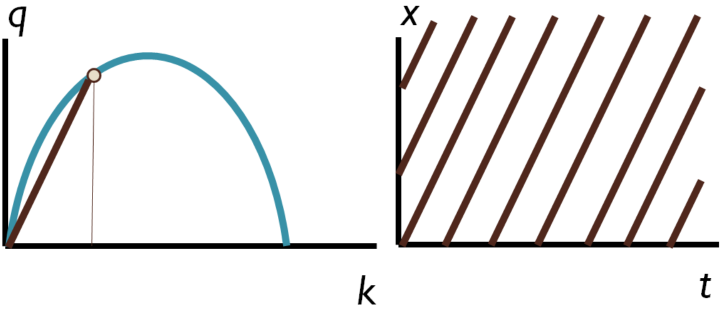


Also notice that there is some point at which flow is maximal. This maximum flow is the capacity q_{max}, which occurs at the critical density k_c.

We can get speed information from the fundamental diagram: $q = uk$, so $u = q/k$.



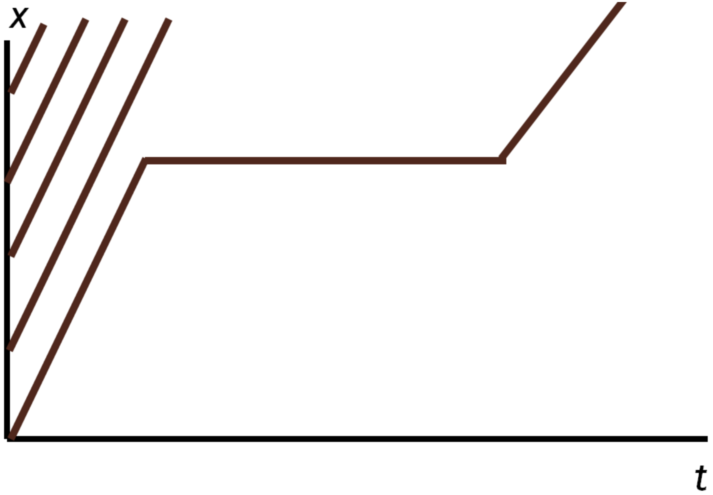
This is the same slope as on a trajectory diagram.

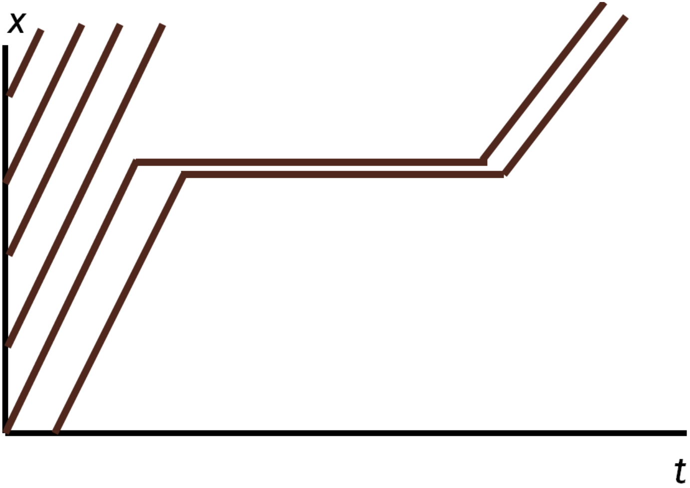


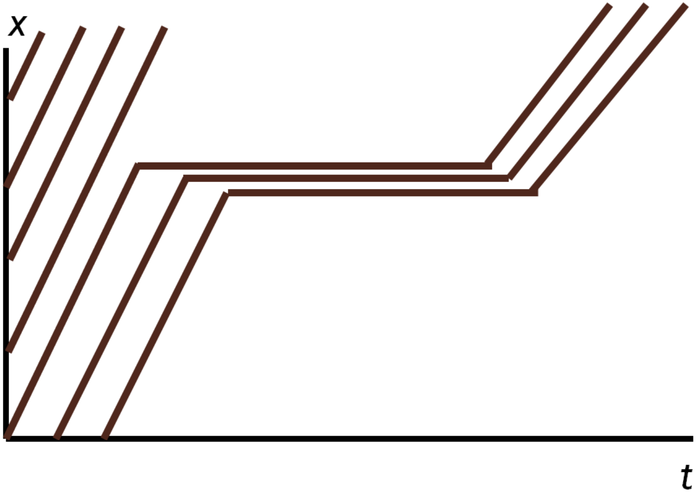
What happens if something interrupts this flow?

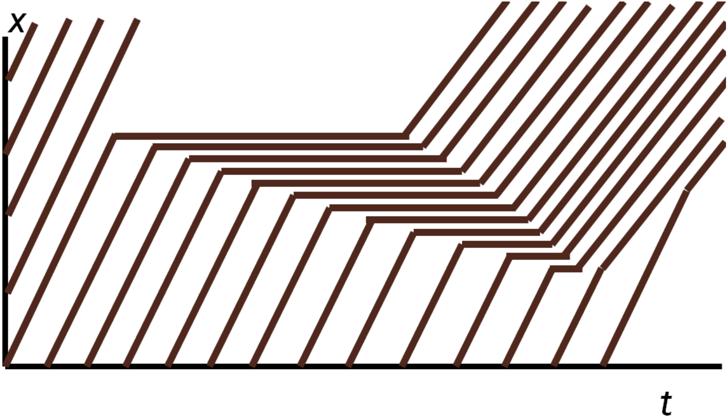
Shockwaves

Let's say one car stops for a while. What happens to the next vehicles?

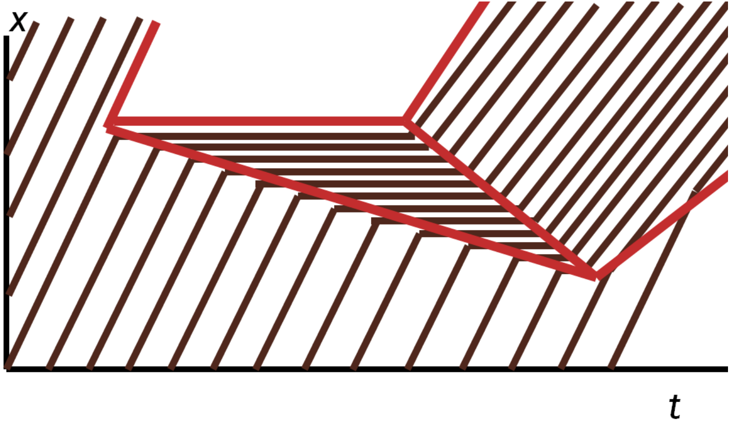



x
t

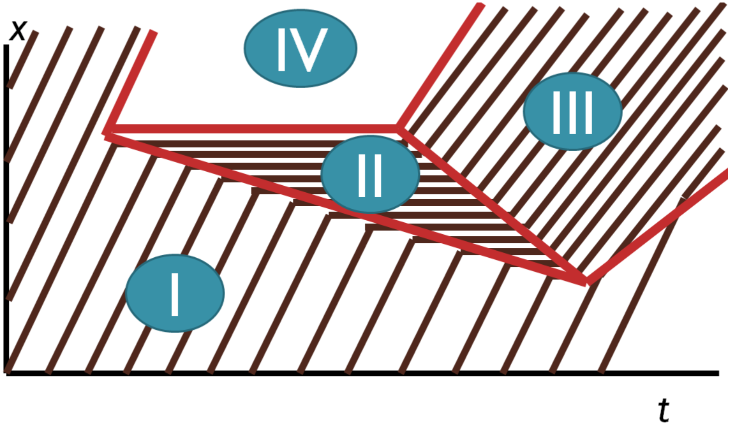

x
t


x
t

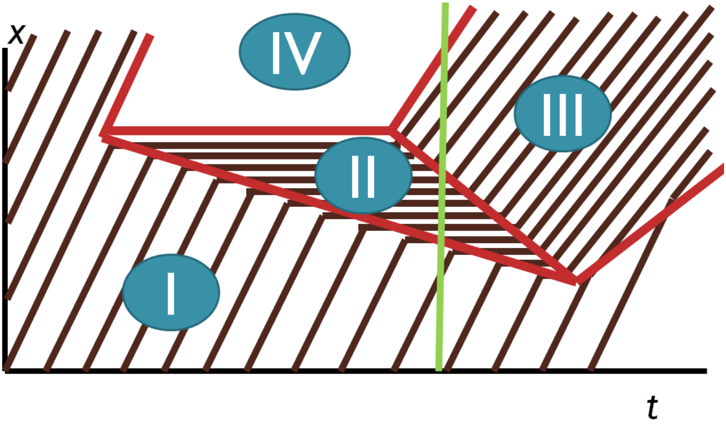
We can identify regions of constant density. The boundaries between these regions are *shockwaves.*

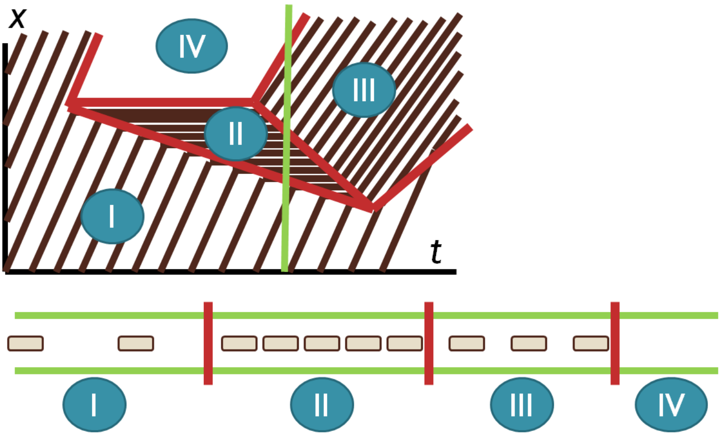


Let's label these regions I, II, III, and IV.



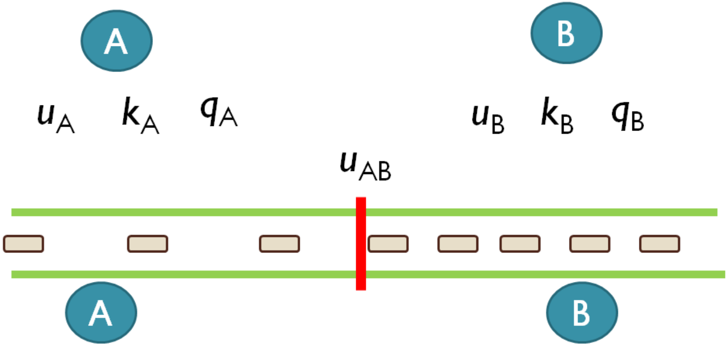
Let's look at a vertical slice of the trajectory diagram.



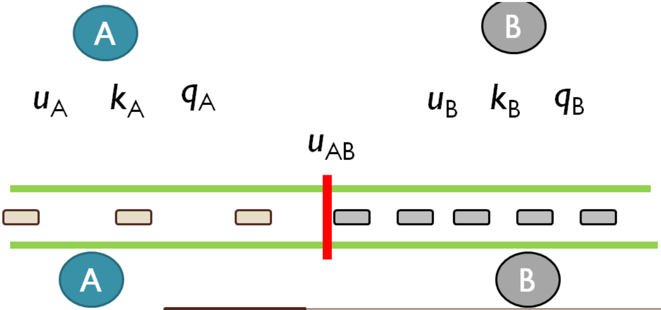


How quickly are these shockwaves moving? (In other words, when will the link fill up?)

Consider an arbitrary shockwave.



What rate are vehicles *entering* the shockwave?

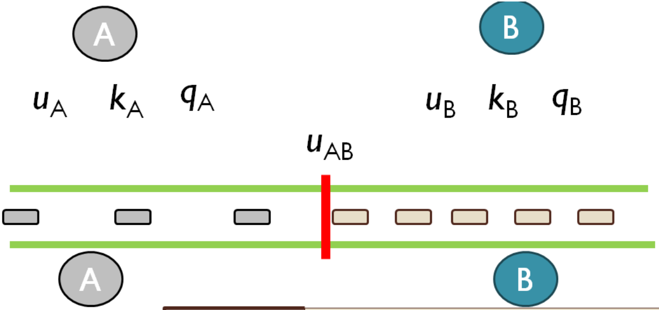


Vehicle speed from the left relative to shockwave

$$q_{AB}^{\rightarrow|} = u_{AB}^{\rightarrow|} k_A$$

$$= (u_A - u_{AB}) k_A$$

What rate are vehicles *leaving* the shockwave?



$$q_{AB}^{|\rightarrow} = u_{AB}^{|\rightarrow} k_A$$
$$= (u_B - u_{AB}) k_B$$

Vehicle speed from the right relative to shockwave

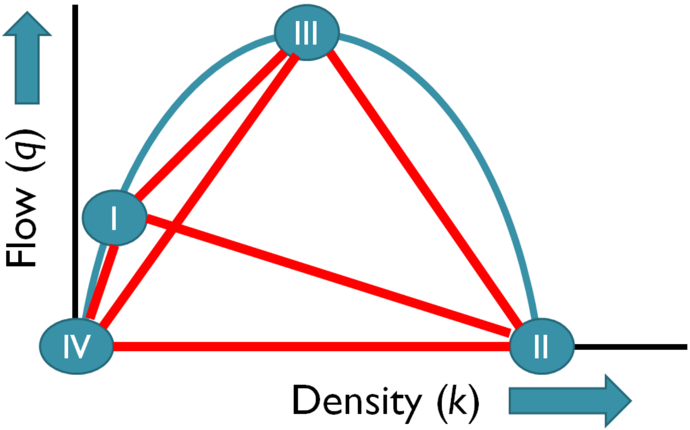
These flow rates should be identical, since vehicles are not appearing or disappearing at the shockwave. Thus

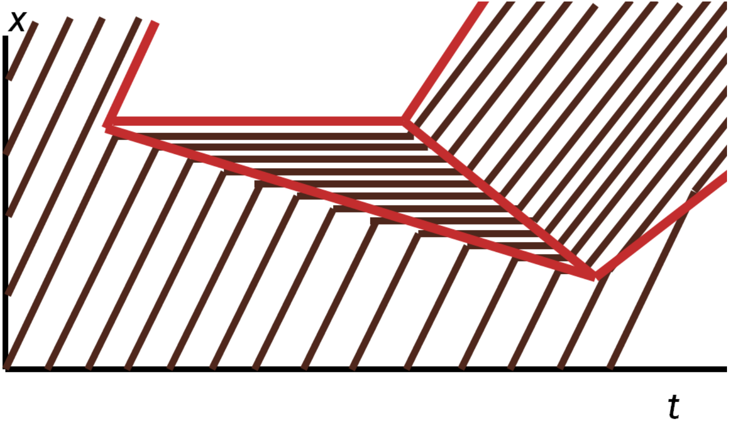
$$(u_A - u_{AB})k_A = (u_B - u_{AB})k_B$$

or

$$u_{AB} = \frac{q_A - q_B}{k_A - k_B}$$

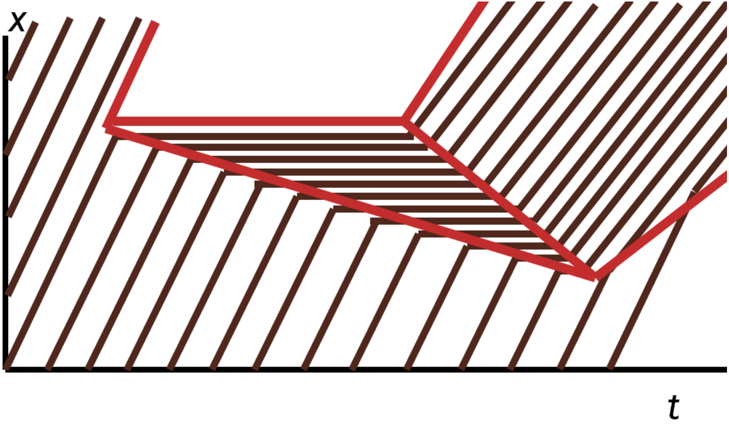
The slope of the shockwaves on a trajectory diagram is the slope of the line connecting the points on the fundamental diagram.




x
t

PDE FORMULATION

In general, the values of q, k, and u can vary with x and t, subject to the fundamental relationship and fundamental diagram. Denote these $q(x,t)$, $k(x,t)$, and $u(x,t)$.



We can also define cumulative counts N as a function of x and t. While actual vehicle trajectories are discrete, we can "smooth" them so $N(x,t)$ is continuous.

Flow and density are related to the cumulative counts:

$$q(x, t) = \frac{\partial N(x, t)}{\partial t}$$

$$k(x, t) = -\frac{\partial N(x, t)}{\partial x}$$

The “solution” to the LWR problem is to find the values $N(x, t)$, $q(x, t)$, and $k(x, t)$ such that

$$q(x, t) = \frac{\partial N(x, t)}{\partial t}$$

$$k(x, t) = -\frac{\partial N(x, t)}{\partial x}$$

$$q(x, t) = Q(k(x, t))$$

given boundary conditions (typically values of N for certain x and t)

Over the next few weeks, we will see several ways to solve this formulation.