Newell's method

CE 391F

January 29, 2013

ANNOUNCEMENTS

- Additional reading: Yperman's dissertation (KU Leuven)

REVIEW

- Flow, speed, and density
- Fundamental relationship, fundamental diagram, trajectory diagram
- Shockwaves

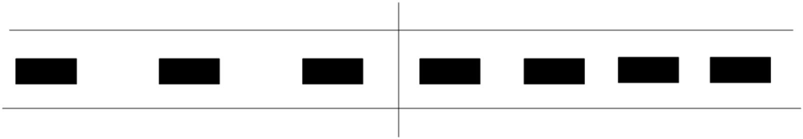
OUTLINE

1. Shockwave example
2. Connection between LWR and cumulative counts
3. Newell's method

SHOCKWAVE EXAMPLES

Flow 1200 veh/hr
Density 20 veh/mi

Flow 1200 veh/hr
Density 60 veh/mi

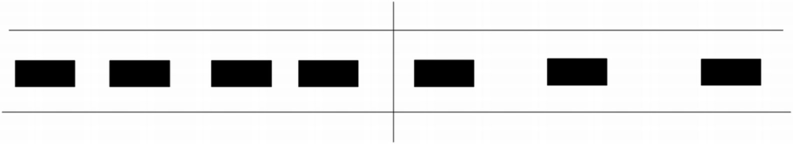


The shockwave between these regions is:

(a) moving upstream.
(b) moving downstream.
(c) stationary.

Flow 1200 veh/hr
Density 60 veh/mi

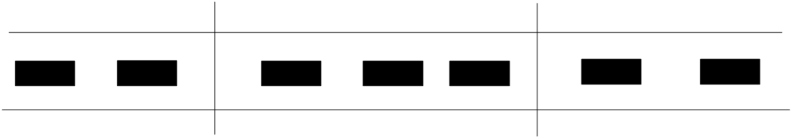
Flow 1600 veh/hr
Density 40 veh/mi



The shockwave between these regions is:

(a) moving upstream.
(b) moving downstream.
(c) stationary.

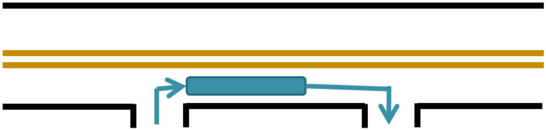
Flow: 1000 veh/hr	600 veh/hr	1800 veh/hr
Density 20 veh/mi	100 veh/mi	40 veh/mi



Will these two shockwaves ever meet?

(a) Yes, upstream of their current locations.
(b) Yes, between their current locations.
(c) Yes, downstream of their current locations.
(d) No.

Example



Roadway has free-flow speed 60 mph, capacity 1800 veh/hr, jam density 120 veh/mi, and fundamental diagram

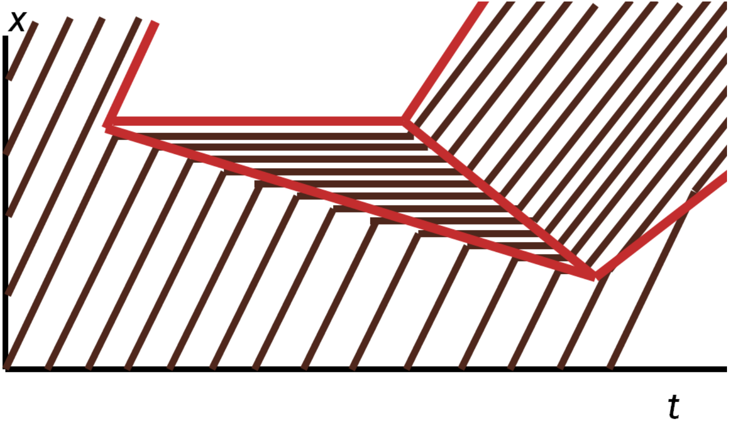
$$q = u_f(k - k^2/k_j)$$

Initially vehicles are traveling at 45 mph. A slow vehicle enters the roadway at 1 PM, driving 20 mph, and turns off 1 mile later.

Identify all regions and shockwaves, and the time at which the last vehicle is driving at 20 mph.

CUMULATIVE COUNTS AND SHOCKWAVES

In general, the values of q, k, and u can vary with x and t, subject to the fundamental relationship and fundamental diagram. Denote these $q(x, t)$, $k(x, t)$, and $u(x, t)$.



We can also define cumulative counts N as a function of x and t. While actual vehicle trajectories are discrete, we can "smooth" them so $N(x, t)$ is continuous.

Flow and density are related to the cumulative counts:

$$q(x,t) = \frac{\partial N(x,t)}{\partial t}$$

$$k(x,t) = -\frac{\partial N(x,t)}{\partial x}$$

If N is twice continuously differentiable,

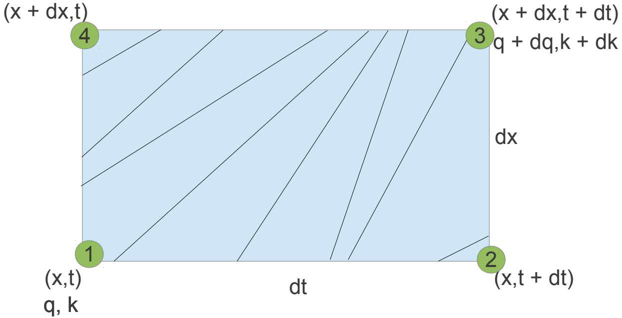
$$\frac{\partial^2 N}{\partial x \partial t} = \frac{\partial^2 N}{\partial t \partial x}$$

Therefore

$$\frac{\partial q}{\partial x} + \frac{\partial k}{\partial t} = 0$$

This is one way to derive the *conservation equation*. This equation holds everywhere except at shockwaves.

Another derivation of the conservation law



1. $N(x,t) = N_0$
2. $N(x, t + dt) = N_0 + q\ dt$
3. $N(x + dx, t + dt) = N_0 + q\ dt - (k + dk)\ dx$
4. $N(x + dx, t) = N_0 + q\ dt - (k + dk)\ dx - (q + dq)\ dt$
5. $N(x,t) = N_0 + q\ dt - (k + dk)\ dx - (q + dq)\ dt + k\ dx$

Therefore $\partial q/\partial x + \partial k/\partial t = 0$

The change in cumulative count number along any curve C is

$$N(x_2, t_2) - N(x_1, t_1) = \int_C q\ dt - k\ dx$$

CHARACTERISTICS

We want to use the conservation law to help solve LWR problems.

In the language of differential equations, we want to find functions $k(x, t)$ and $q(x, t)$ such that

1. Conservation is satisfied: $\partial q / \partial x + \partial k / \partial t = 0$
2. The fundamental diagram is satisfied: $q(x, t) = Q(k(x, t))$
3. Any boundary conditions are satisfied.

Luckily, these PDEs can usually be solved without too much difficulty.

The fundamental diagram implies that q is a function of k.

Therefore, the conservation law $\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0$ can be rewritten

$$\frac{\partial k}{\partial t} + \frac{dq}{dk}\frac{\partial k}{\partial x} = 0$$

which is a PDE in $k(x, t)$ alone.

We can solve this PDE by looking for "characteristics," curves along which $k(x, t)$ is constant.

$$\frac{\partial k}{\partial t} + \frac{dq}{dk}\frac{\partial k}{\partial x} = 0$$

In this model, the characteristics are straight lines. Consider a line with slope dq/dk, so $dx = \frac{dq}{dk}dt$

If we move in this direction,

$$dk = \frac{\partial k}{\partial t}dt + \frac{\partial k}{\partial x}dx$$

$$dk = -\frac{dq}{dk}\frac{\partial k}{\partial x}dt + \frac{\partial k}{\partial x}\frac{dq}{dk}dt = 0$$

so $k(x,t)$ is constant along a line with slope dq/dk. This is called the *wave speed*.

So, if I know the value of $k(x,t)$ at any point (e.g., boundary condition), I know the value of $k(x,t)$ at all points along the line through (x,t) with slope $dq(x,t)/dk$...unless there is "something else" which interferes (i.e., a shockwave).

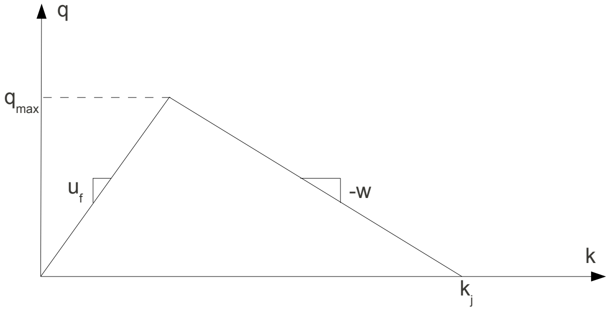
In the "uncongested" portion of the fundamental diagram, $dq/dk > 0$, so **uncongested states propagate downstream**.

In the "congested" portion of the fundamental diagram, $dq/dk < 0$, so **congested states propagate upstream**.

In other words, where there is no congestion, upstream conditions prevail. Where congestion exists, downstream conditions prevail.

NEWELL’S METHOD

Newell's method is an easier alternative to solving the LWR model.



The main feature is a simplified fundamental diagram with only two wave speeds: u_f for the uncongested portion, and $-w$ for the congested portion.

Notice that in Newell's model, speed does not drop until density exceeds the critical density and congestion sets in.

The rough logic behind Newell's method:

1. We want to calculate $k(x,t)$ or $N(x,t)$ at some point (x,t).
2. Either this point is congested or uncongested.
3. If congested, the wave speed is $-w$, so past conditions downstream will determine $k(x,t)$ and $N(x,t)$ here.
4. If uncongested, the wave speed is u_f, so past conditions downstream will determine $k(x,t)$ and $N(x,t)$ here.
5. Of these two possibilities, the correct solution is the one corresponding to the *lowest* $N(x,t)$ value.

If upstream conditions prevail, the $N(x,t)$ value based on the uncongested wave speed will be lower. If downstream conditions prevail, the $N(x,t)$ value based on the congested wave speed will be lower.

The major tools in Newell's method:

1. $$N(x_2, t_2) - N(x_1, t_1) = \int_C q\ dt - k\ dx$$
2. k (and therefore q) is constant along characteristics
3. Characteristics are straight lines, so $q\ dt - k\ dx$ is *constant*, so the integral is easy to evaluate.
4. With the simplified fundamental diagram, there are only two characteristic slopes possible

Change in vehicles along characteristic with positive slope

These characteristics reflect uncongested conditions, and have slope (wave speed) u_f.

$$\int_C q\ dt - k\ dx = \int_C \left(q - k\frac{dq}{dk}\right)\ dt$$

dq/dk is the wave speed, u_f.

However, u_f is also the traffic speed for the uncongested case, so $q = u_f k$ and

$$\int_C \left(q - k\frac{dq}{dk}\right)\ dt = 0$$

$N(x, t)$ is constant along forward-moving characteristics. In other words, if you move with the speed of uncongested traffic, you should observe no change in the cumulative vehicle count.

Change in vehicles along characteristic with negative slope

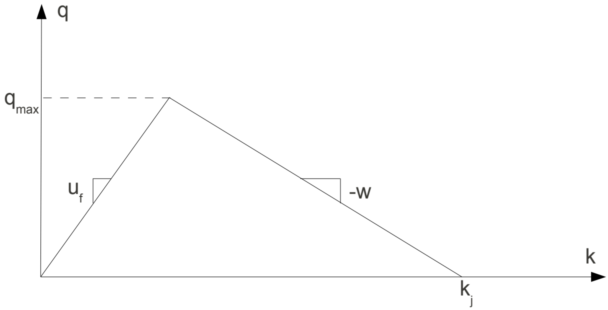
These characteristics reflect uncongested conditions, and have slope (wave speed) $-w$.

$$\int_C q\ dt - k\ dx = \int_C \left(\frac{q}{dq/dk} - k \right)\ dx$$

dq/dk is the wave speed, $-w$.

$$\int_C q\ dt - k\ dx = -\int_C (k + q/w)\ dx$$

From the fundamental diagram, $k + q/w = k_j$



So

$$\int_C q \ dt - k \ dx = -\int_C k_j dx = k_j(x_2 - x_1)$$

If you are moving at the backward wave speed, the cumulative vehicle count increases at the rate of the jam density.

Example

A link is 1 mile long, and has free flow speed 30 mph, backward wave speed 15 mph, and jam density 200 veh/mi. Vehicles enter upstream at a rate of 1200 veh/hr. Three minutes from now, the downstream traffic signal turns red. Four minutes from now, what is the cumulative count value at the midpoint of the link? Has the queue reached this point?