Daganzo's variational method

CE 391F

January 31, 2013

ANNOUNCEMENTS

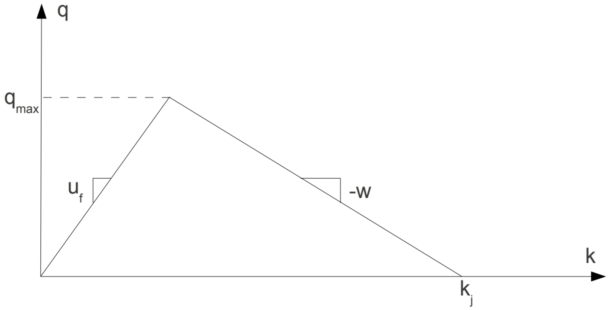
- Homework 1 to be assigned Tuesday
- Suggested reading: Daganzo (2005a,b)

REVIEW

- Fundamental relationship, fundamental diagram, trajectory diagram
- Shockwaves
- Newell's method

NEWELL'S METHOD

Newell's method is an easier alternative to solving the LWR model.



The main feature is a simplified fundamental diagram with only two wave speeds: u_f for the uncongested portion, and $-w$ for the congested portion.

Notice that in Newell's model, speed does not drop until density exceeds the critical density and congestion sets in.

The rough logic behind Newell's method:

1. We want to calculate $k(x,t)$ or $N(x,t)$ at some point (x,t).
2. Either this point is congested or uncongested.
3. If congested, the wave speed is $-w$, so past conditions downstream will determine $k(x,t)$ and $N(x,t)$ here.
4. If uncongested, the wave speed is u_f, so past conditions downstream will determine $k(x,t)$ and $N(x,t)$ here.
5. Of these two possibilities, the correct solution is the one corresponding to the *lowest* $N(x,t)$ value.

If upstream conditions prevail, the $N(x,t)$ value based on the uncongested wave speed will be lower. If downstream conditions prevail, the $N(x,t)$ value based on the congested wave speed will be lower.

The major tools in Newell's method:

1. $$N(x_2, t_2) - N(x_1, t_1) = \int_C q \; dt - k \; dx$$
2. k (and therefore q) is constant along characteristics
3. Characteristics are straight lines, so $q \; dt - k \; dx$ is *constant*, so the integral is easy to evaluate.
4. With the simplified fundamental diagram, there are only two characteristic slopes possible

Change in vehicles along characteristic with positive slope

These characteristics reflect uncongested conditions, and have slope (wave speed) u_f.

$$\int_C q \; dt - k \; dx = \int_C \left(q - k \frac{dq}{dk} \right) \; dt$$

dq/dk is the wave speed, u_f.

However, u_f is also the traffic speed for the uncongested case, so $q = u_f k$ and

$$\int_C \left(q - k \frac{dQ}{dk} \right) \; dt = 0$$

$N(x,t)$ is constant along forward-moving characteristics. In other words, if you move with the speed of uncongested traffic, you should observe no change in the cumulative vehicle count.

Change in vehicles along characteristic with negative slope

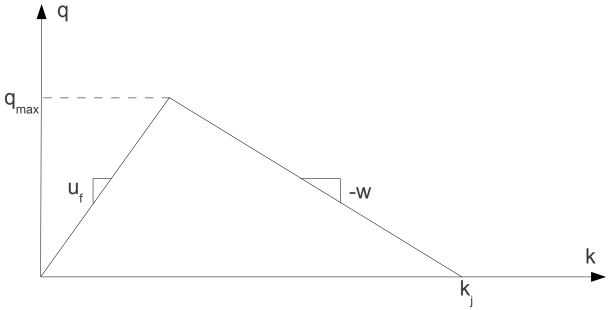
These characteristics reflect uncongested conditions, and have slope (wave speed) $-w$.

$$\int_C q\ dt - k\ dx = \int_C \left(\frac{q}{dQ/dk} - k \right)\ dx$$

dQ/dk is the wave speed, $-w$.

$$\int_C q\ dt - k\ dx = -\int_C (k + q/w)\ dx$$

From the fundamental diagram, $k + q/w = k_j$



So

$$\int_C q \ dt - k \ dx = -\int_C k_j dx = k_j(x_2 - x_1)$$

If you are moving at the backward wave speed, the cumulative vehicle count increases at the rate of the jam density.

Example

A link is 1 mile long, and has free flow speed 30 mph, backward wave speed 15 mph, and jam density 200 veh/mi. Vehicles enter upstream at a rate of 1200 veh/hr. Three minutes from now, the downstream traffic signal turns red. Four minutes from now, what is the cumulative count value at the midpoint of the link? Has the queue reached this point?

OUTLINE

1. Limitations of Newell's method
2. Generalizing the set of paths considered
3. A variational approach
4. Solving continuum shortest path problems

GENERALIZING NEWELL'S METHOD

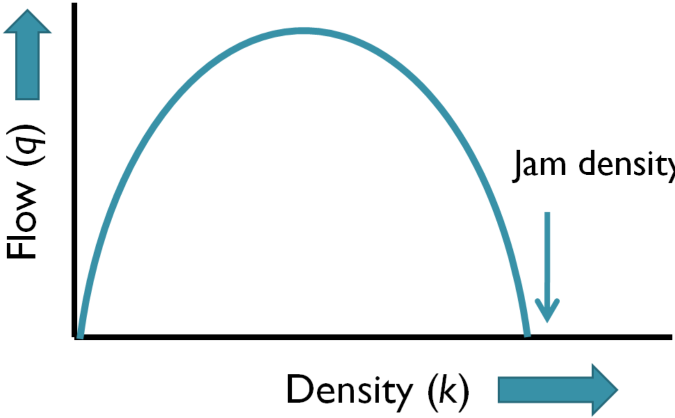
What would happen in Newell's method if the fundamental diagram varies with space or time?

Recall characteristics satisfy the equation

$$\frac{\partial k}{\partial t} + \frac{dq}{dk}\frac{\partial k}{\partial x} = 0$$

As before, we have $dx = \dfrac{dQ(x,t)}{dk(x,t)}dt$, but this is no longer a straight line. Furthermore, the density may vary along characteristics (in a predictable way, but still makes our lives more complicated).

How would you apply Newell's method if the fundamental diagram is smooth?



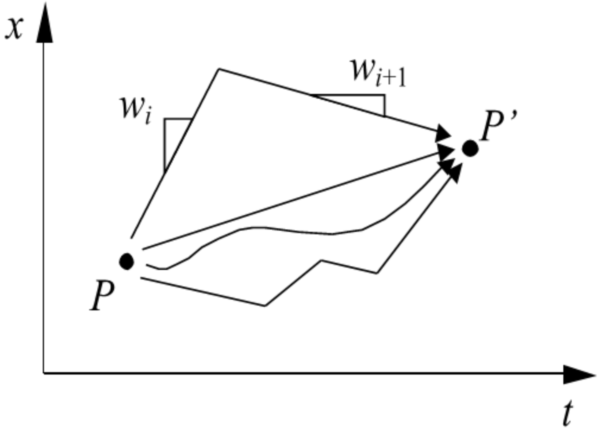
So, Newell's method has several limitations:

- Difficult to handle problems where fundamental diagram varies with space or time
- What if the fundamental diagram is not piecewise linear?

In 2005, Daganzo proposed a variational approach which generalizes Newell's method.

A VARIATIONAL APPROACH

The main idea in Daganzo's method is to expand the set of paths considered.



Rather than just characteristics ("wave paths"), we will now consider *any* "valid path" as potentially giving the right change in cumulative count.

Valid paths are:

- Always moving in the direction of increasing time
- Have a speed within the range of possible wave speeds
- Piecewise differentiable

For the triangular fundamental diagram, the range of wave speeds is $[-w, u_f]$.

Remember how we derived Newell's method for wave paths:

1. $$N(x_2, t_2) - N(x_1, t_1) = \int_C q\ dt - k\ dx$$
2. k (and therefore q) is constant along characteristics
3. Characteristics are straight lines, so $q\ dt - k\ dx$ is *constant*, so the integral is easy to evaluate.
4. With the simplified fundamental diagram, there are only two characteristic slopes possible

Can we do something similar for valid paths where the fundamental diagram is inhomogeneous?

We still have $N(x_2, t_2) - N(x_1, t_1) = \int_C q\ dt - k\ dx$, but we don't know how k varies along C.

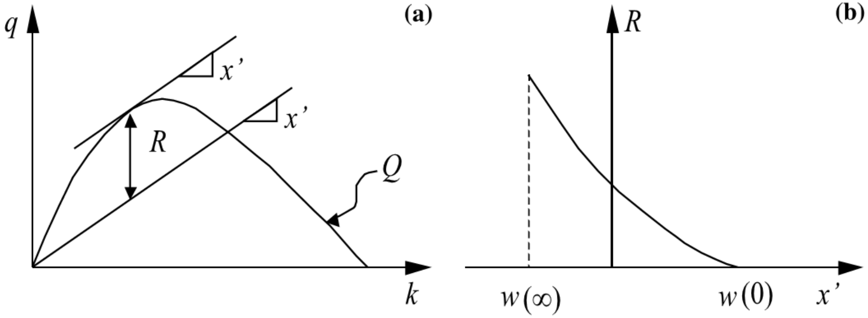
Daganzo's idea is to define $\Delta(P)$ to be the *maximum possible* change in N between the endpoints of a path P

Consider any point (x, t) on path P, and let dx/dt be the slope of the path at this point. What is the maximum possible value of $q\ dt - k\ dx$?

We don't know the density k, but if we did the rate of change would be $Q(k, x, t)\ dt - k\ dx = (Q(k, x, t) - k\ dx/dt)dt$

So we want to find $\sup_k\{Q(k, x, t) - k\ dx/dt\}$. Call this value $R(x, t, dx/dt)$.

$R(x, t, dx/dt)$ does *not* depend on the boundary conditions, or any N values... it only depends on the fundamental diagram at the point (x, t) and the speed dx/dt.


(a)
q
x'
x'
R
Q
k
(b)
R
w(∞)
w(0)
x'

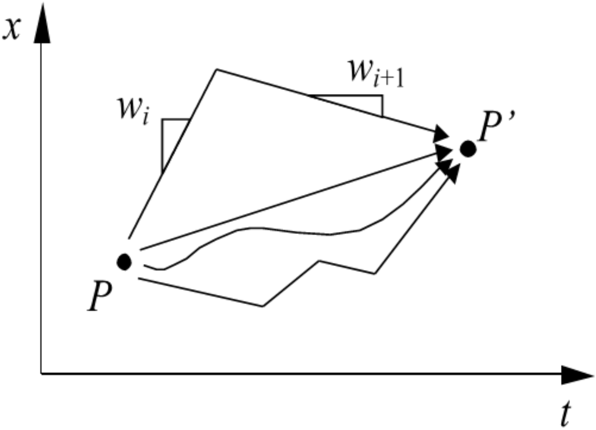
Therefore, the maximum change in vehicle number between the endpoints of path P is $\Delta(P) = \int_C R(x, t, dx/dt)\ dt$

Therefore, $N(x_2, t_2) - N(x_1, t_1) \leq \Delta(P)$ for every valid path P connecting (x_1, t_1) to (x_2, t_2).

In fact, the actual value of $N(x, t)$ is the minimum value of $\Delta(P)$ across all valid paths P starting at a point where N is known (e.g., a boundary condition) and ending at (x, t). (Proof: Daganzo, 2005)

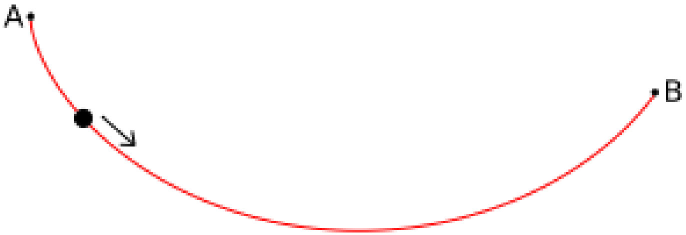
Newell's method can be recovered from Daganzo's: if the fundamental diagram is triangular, and identical at every location in time, the valid path P achieving this minimum is a straight line at the wave speed.

How can we find the *valid path* P with the minimum value of $\Delta(P)$?



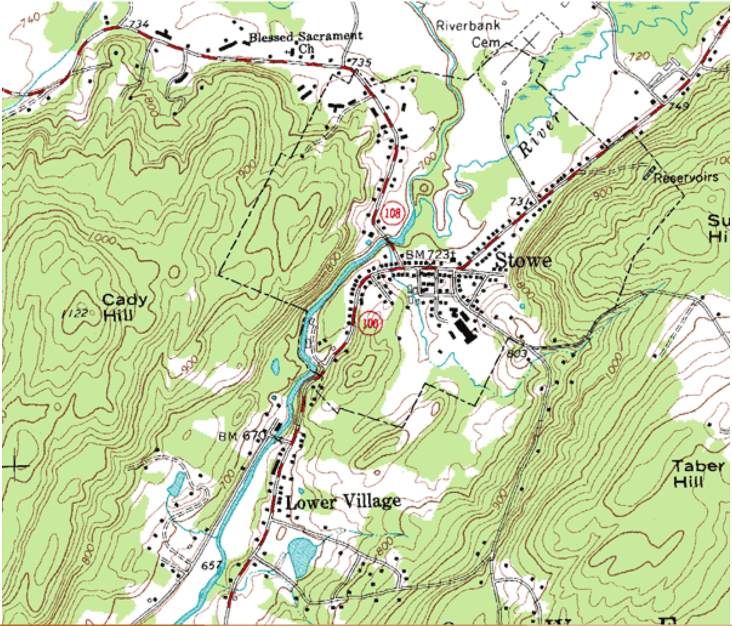
This is the type of problem solved using the *calculus of variations*, where we try to find *functions* maximizing or minimizing some objective. Some examples...

The brachistochrone problem



Given two points A and B in a vertical plane where gravity is the only force, find the curve which minimizes the time needed for an object to fall from A to B.

Minimum-detection paths for military path-finding

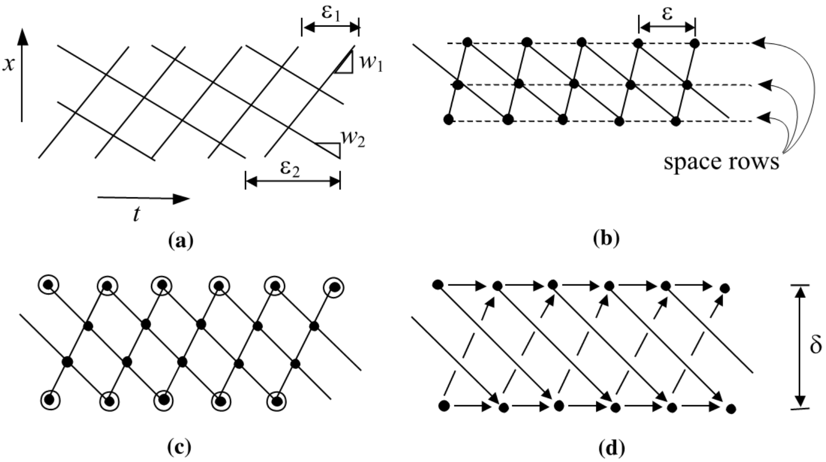


CONTINUUM SHORTEST PATH PROBLEMS

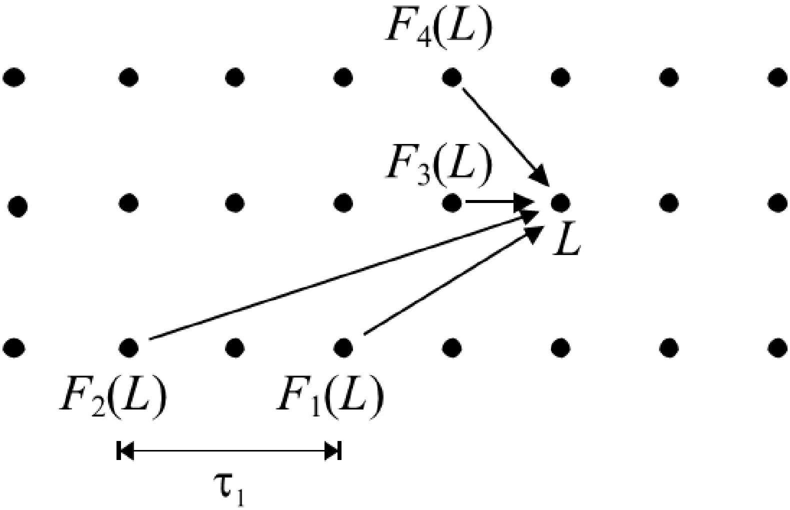
This type of problem is generally solved by overlaying a network of straight lines onto the space-time diagram.

The cost of each link is given by $R(x, t, dx/dt)\delta t$, where δt is the temporal extent of the link.

We can now solve a regular shortest path problem on this network to identify the valid path minimizing $\Delta(P)$.



ε_1
x
w_1
w_2
ε_2
t
(a)
ε
space rows
(b)
(c)
δ
(d)


$F_4(L)$
$F_3(L)$
L
$F_2(L)$
$F_1(L)$
τ_1

By choosing the network properly, we can limit the error from this approximation to the continuum.

A pair of nodes is *valid* if there is some valid continuum path P connecting them.

A network is *validly connected* if every valid pair of nodes is connected in the network.

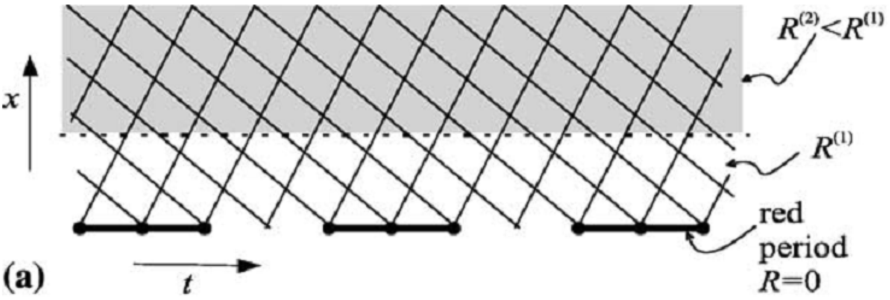
A network is *sufficient* if the shortest path in the network between every valid pair of nodes is in fact the optimal continuum path.

A network is *z-sufficient* if the shortest network path is within z of the optimal continuum path.

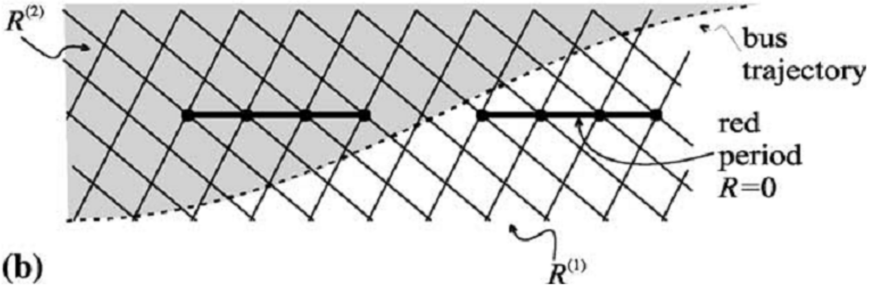
For homogeneous problems with triangular fundamental diagrams, networks formed by equidistant lines with slopes u_f and $-w$ are sufficient.

For piecewise-linear fundamental diagrams with more than two pieces, we can find z-sufficient networks.

Inhomogeneous problems can often be handled by adding "shortcuts" or by changing link costs in different regions.



$R^{(2)} < R^{(1)}$
x
$R^{(1)}$
red period $R=0$
(a)
t



$R^{(2)}$
bus
trajectory
red
period
$R=0$
$R^{(1)}$
(b)