Daganzo's variational method

CE 391F

February 5, 2013

ANNOUNCEMENTS

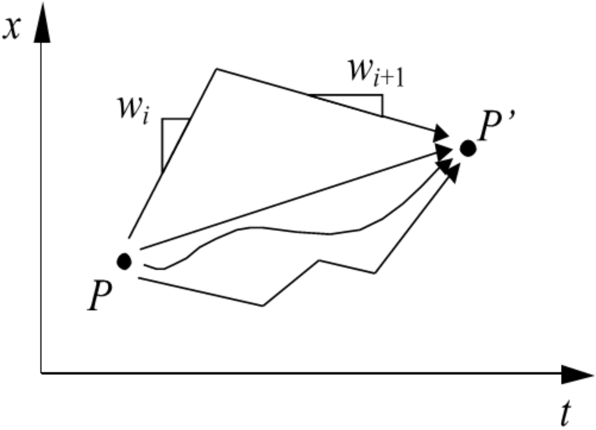
- Homework 1 online
- Suggested reading: Daganzo (2005a,b); Daganzo (1994,1995)

REVIEW

- Newell's method and its limitations
- Calculus of variations

DAGANZO’S METHOD

The main idea in Daganzo's method is to expand the set of paths considered.



Rather than just characteristics ("wave paths"), we will now consider *any* "valid path" as potentially giving the right change in cumulative count.

Valid paths are:

- Always moving in the direction of increasing time
- Have a speed within the range of possible wave speeds
- Piecewise differentiable

For the triangular fundamental diagram, the range of wave speeds is $[-w, u_f]$.

Remember how we derived Newell's method for wave paths:

1. $$N(x_2, t_2) - N(x_1, t_1) = \int_C q\ dt - k\ dx$$
2. k (and therefore q) is constant along characteristics
3. Characteristics are straight lines, so $q\ dt - k\ dx$ is *constant*, so the integral is easy to evaluate.
4. With the simplified fundamental diagram, there are only two characteristic slopes possible

Can we do something similar for valid paths where the fundamental diagram is inhomogeneous?

We still have $N(x_2, t_2) - N(x_1, t_1) = \int_C q\ dt - k\ dx$, but we don't know how k varies along C.

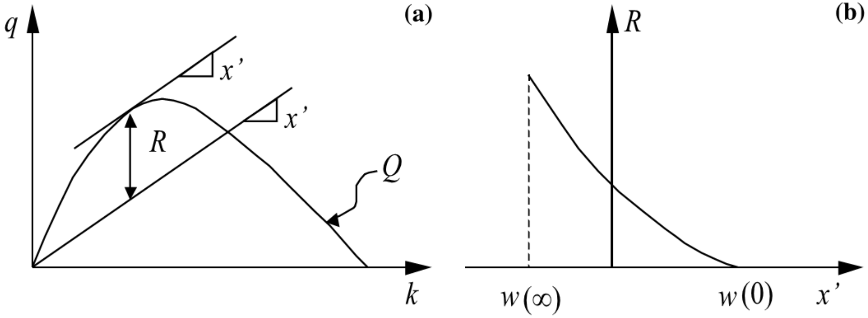
Daganzo's idea is to define $\Delta(P)$ to be the *maximum possible* change in N between the endpoints of a path P

Consider any point (x, t) on path P, and let dx/dt be the slope of the path at this point. What is the maximum possible value of $q\ dt - k\ dx$?

We don't know the density k, but if we did the rate of change would be $Q(k, x, t)\ dt - k\ dx = (Q(k, x, t) - k\ dx/dt)dt$

So we want to find $\sup_k\{Q(k, x, t) - k\ dx/dt\}$. Call this value $R(x, t, dx/dt)$.

$R(x, t, dx/dt)$ does *not* depend on the boundary conditions, or any N values... it only depends on the fundamental diagram at the point (x, t) and the speed dx/dt.


(a)
q
R
x'
x'
Q
k
(b)
R
w(∞)
w(0)
x'

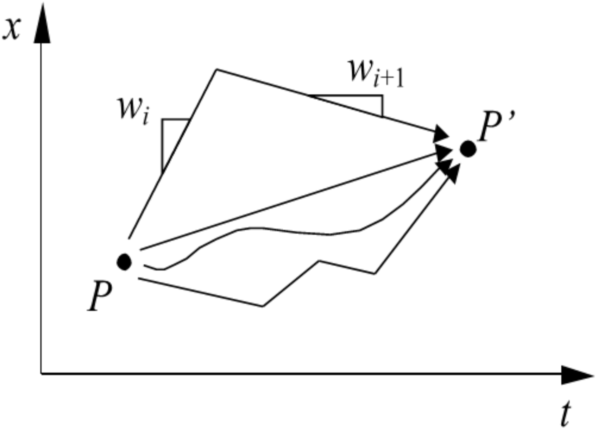
Therefore, the maximum change in vehicle number between the endpoints of path P is $\Delta(P) = \int_C R(x, t, dx/dt)\ dt$

Therefore, $N(x_2, t_2) - N(x_1, t_1) \leq \Delta(P)$ for every valid path P connecting (x_1, t_1) to (x_2, t_2).

In fact, the actual value of $N(x, t)$ is the minimum value of $\Delta(P)$ across all valid paths P starting at a point where N is known (e.g., a boundary condition) and ending at (x, t). (Proof: Daganzo, 2005)

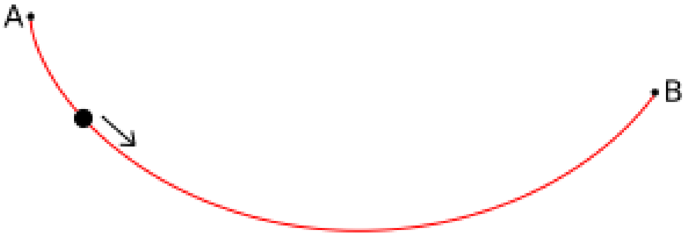
Newell's method can be recovered from Daganzo's: if the fundamental diagram is triangular, and identical at every location in time, the valid path P achieving this minimum is a straight line at the wave speed.

How can we find the *valid path* P with the minimum value of $\Delta(P)$?



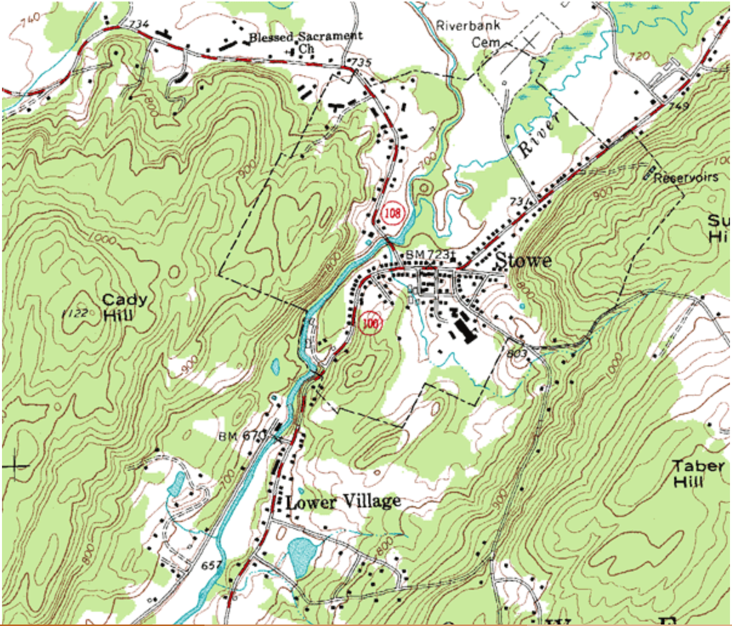
This is the type of problem solved using the *calculus of variations*, where we try to find *functions* maximizing or minimizing some objective. Some examples...

The brachistochrone problem



Given two points A and B in a vertical plane where gravity is the only force, find the curve which minimizes the time needed for an object to fall from A to B.

Minimum-detection paths for military path-finding

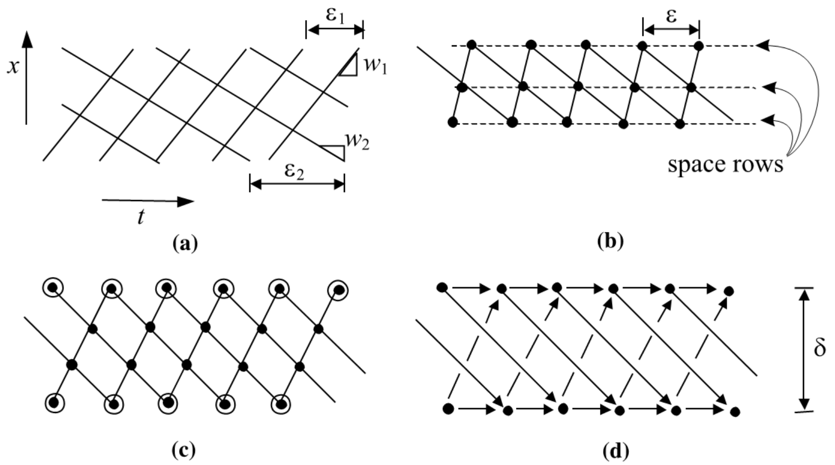


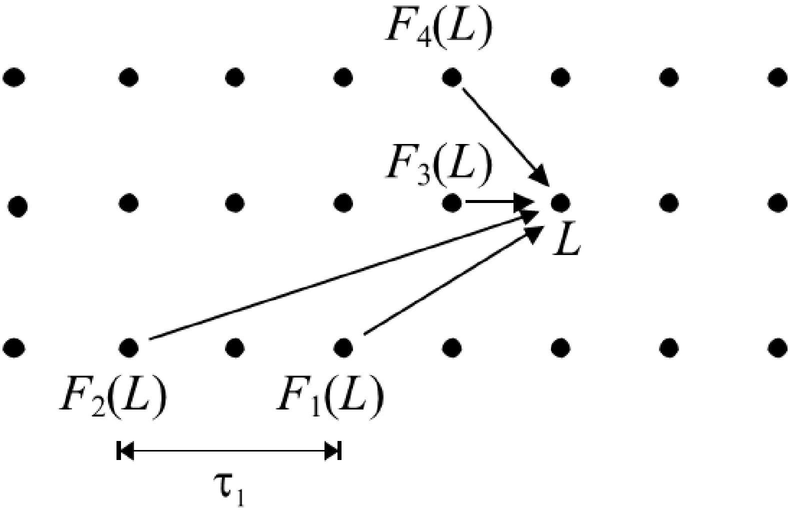
CONTINUUM SHORTEST PATH PROBLEMS

This type of problem is generally solved by overlaying a network of straight lines onto the space-time diagram.

The cost of each link is given by $R(x, t, dx/dt)\delta t$, where δt is the temporal extent of the link.

We can now solve a regular shortest path problem on this network to identify the valid path minimizing $\Delta(P)$.


ε1
x
w1
w2
ε2
t
(a)
ε
space rows
(b)
(c)
δ
(d)


$F_4(L)$
$F_3(L)$
L
$F_2(L)$
$F_1(L)$
τ_1

By choosing the network properly, we can limit the error from this approximation to the continuum.

A pair of nodes is *valid* if there is some valid continuum path P connecting them.

A network is *validly connected* if every valid pair of nodes is connected in the network.

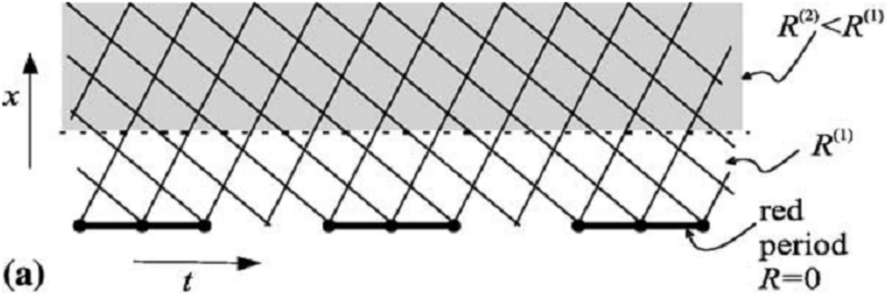
A network is *sufficient* if the shortest path in the network between every valid pair of nodes is in fact the optimal continuum path.

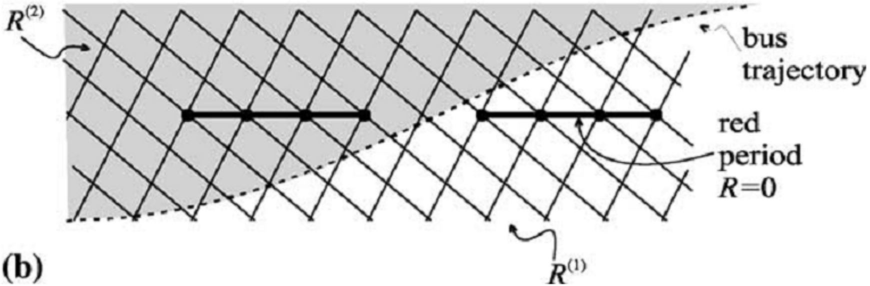
A network is *z-sufficient* if the shortest network path is within z of the optimal continuum path.

For homogeneous problems with triangular fundamental diagrams, networks formed by equidistant lines with slopes u_f and $-w$ are sufficient.

For piecewise-linear fundamental diagrams with more than two pieces, we can find z-sufficient networks.

Inhomogeneous problems can often be handled by adding “shortcuts” or by changing link costs in different regions.


$R^{(2)}<R^{(1)}$
x
$R^{(1)}$
red
period
$R=0$
(a)
t



$R^{(2)}$
bus trajectory
red period $R=0$
$R^{(1)}$
(b)

EXAMPLES

Consider a single-lane roadway with a triangular fundamental diagram, a free-flow speed of 60 mi/hr, a backward wave speed of 30 mi/hr, and a jam density of 180 veh/mi. Initially the roadway is empty; then for at each time $t \in [0, 30]$, vehicles enter at a rate of $2t$ vehicles per minute, after which vehicles enter at a constant rate of 60 veh/min.

A slow-moving truck enters the roadway one mile downstream at time $t = 0$, and travels at 30 mi/hr. No vehicles can pass. This vehicle turns off of the roadway one hour later.

A slow-moving truck enters the roadway one mile downstream at time $t = 0$, and travels at 30 mi/hr, but vehicles can pass at a rate of 10 veh/min. This vehicle turns off of the roadway one hour later.