CE 391F: Homework 1
Solutions

Problem 1. *Consider the following five sets of ten speed observations made as vehicles passing a fixed point:*

50	*50*	*50*	*50*	*50*	*50*	*50*	*50*	*50*
51	*49*	*52*	*54*	*55*	*47*	*46*	*44*	*52*
57	*46*	*58*	*44*	*49*	*52*	*47*	*44*	*53*
40	*55*	*55*	*37*	*40*	*55*	*60*	*53*	*55*
38	*62*	*50*	*35*	*75*	*40*	*45*	*50*	*55*

For each data set, calculate the time-mean and space-mean speeds. Find a relationship between the space-mean speed, time-mean speed, and some measure of variability of the data set (such as standard deviation or variance).

The five data sets all have time-mean speeds of 50, and space-mean speeds of 50, 49.7, 49.5, 48.6, and 47.4. The difference between the space-mean speed and time-mean speed is roughly proportional to the variance in the time-mean data; for the best-fit line, the constant of proportionality is slightly larger than the time-mean speed.

Problem 2. *Initially, traffic is flowing at 45 mph on a roadway with the fundamental diagram* $q = 60(k - k^2/120)$ *with* k *expressed in vehicles per mile. At 1 PM, a slow-moving vehicle turns onto the roadway at milepost 10, driving at 10 mph. However, vehicles can occasionally pass the slow-moving vehicle, resulting in an average vehicle speed of 20 mph just upstream of the slow-moving vehicle. This vehicle turns off the roadway at milepost 20; at this point, vehicles which had to slow down now begin to move at the maximum flow rate (capacity). Use shockwave theory to diagram all the shockwaves created by this event on a space-time diagram. Label the speed of each shockwave and the space-time coordinates of each point where shockwaves intersect (assuming that the traffic is moving in the direction of increasing milepost). Draw a few representative vehicle trajectories; in particular, draw at least one trajectory which crosses as many regions in the shockwave diagram as possible. (Your diagram should be roughly to scale.)*

With this fundamental diagram, the speed-density relationship is $u = q/k = 60(1 - k/120)$. Region I will consist of the initial vehicle flow, where we are given $u_1 = 45$ mph; from the speed-density relationship, we then have $k_1 = 30$ veh/mi, and therefore $q_1 = u_1 k_1 = 1350$ veh/hr. Immediately upstream of the slow-moving vehicle, we have Region II of traffic moving at a reduced speed: we are given $u_2 = 20$ mph, therefore $k_2 = 80$ veh/mi and $q_2 = 1600$ veh/hr. We are not told the speed, density, or flow of the vehicles that have passed the slow-moving vehicle and are immediately downstream (Region III); however, the speed of the shockwave between Regions II and III must be 10 mph[1]. Thus $u_{23} = \frac{q_3 - q_2}{k_3 - k_2} = 10$ mph. Substituting known

[1] If you aren't convinced that the shockwave trajectory must match that of the slow-moving vehicle, consider this logic: by flow conservation, the rate of vehicles passing the slow-moving vehicle as they leave Region II must be the same as the rate which they enter Region III. Expressing these flow rates based on relative velocity and equating them gives you the equation for shockwave velocity.

values, and then the fundamental diagram, we have:

$$
\begin{aligned}
10 = \frac{q_3 - 1600}{k_3 - 80} &\iff 10k_3 - 800 = q_3 \\
&\iff 10k_3 - 800 = 60(k_3 - k_3^2/120) \\
&\iff k_3^2 - 100k_3 - 1600 = 0 \\
&\iff (k_3 - 80)(k_3 - 20) = 0
\end{aligned}
$$

The two solutions to this quadratic equation are $k_3 = 80$ and $k_3 = 20$; the physically meaningful solution is $k_3 = 20$ veh/mi. (Otherwise, $k_3 = k_2$ and there is no shockwave; equivalently, flow would remain in the congested state even after passing the slow-moving vehicle, which would not occur.) Therefore $u_3 = 50$ mph and $q_3 = 1000$ veh/mi. In Region IV after the slow-moving vehicle leaves we are told that q_4 is the capacity; with a parabolic fundamental diagram, this maximum occurs at half the jam density, so $k_4 = 60$ veh/mi, $u_4 = 30$ mph, and $q_4 = 1800$ veh/hr. Furthermore, upon sketching a trajectory diagram (Figure 1), we see there is an empty Region V between the slower-moving flow in Region IV, and the faster-moving flow in Region III. We calculate the speeds of eaach shockwave as follows:

Shockwave I-II: $u_{12} = (q_2 - q_1)/(k_2 - k_1) = 250/50 = 5$ mph downstream, beginning at milepost 10 at 1 PM

Shockwave I-III: $u_{13} = 350/10 = 35$ mph downstream, beginning at milepost 10 at 1 PM

Shockwave I-IV: $u_{14} = 450/30 = 15$ mph downstream. This shockwave appears starting at the place and time when shockwaves I-II and II-IV meet.

Shockwave I-V: $u_{15} = 1350/30 = 45$ mph downstream. This shockwave starts at the place and time where shockwaves I-III and III-V meet.

Shockwave II-III: $u_{23} = 600/60 = 10$ mph downstream, beginning at milepost 10 at 1 PM

Shockwave II-IV: $u_{24} = -200/20 = 10$ mph upstream, beginning at milepost 20 at 2 PM

Shockwave III-V: $u_{35} = 1000/20 = 50$ mph downstream, beginning at milepost 20 at 2 PM

Shockwave IV-V: $u_{45} = 1800/60 = 30$ mph downstream, beginning at milepost 20 at 2 PM

The following shockwaves intersect:

- Shockwaves I-II, I-III, and II-III all meet at milepost 10 at 1 PM (when they all begin)
- Shockwaves II-III, II-IV, III-V, and IV-V meet at milepost 20 at 2 PM (II-III ends here, the rest begin)
- Shockwaves I-II and II-IV meet at the same point shockwaves I-IV begins. To find this intersection point, note that I-II begins at milepost 10 at 1 PM and moves 5 mph downstream, while II-IV begins at milepost 20 at 2 PM and moves 10 mph upstream. Using simple algebra they intersect at milepost $16\frac{2}{3}$ at 2:20.

- Shockwaves I-III and III-V eventually meet at the same point where the shockwave I-V begins. I-III begins at milepost 10 at 1 PM and moves 35 mph downstream, while III-V begins at milepost 20 at 2 PM and moves 50 mph downstream; they meet at milepost $103\frac{1}{3}$ at 3:40 PM.

Problem 3. Consider a long roadway segment with a triangular fundamental diagram with free-flow speed 60 mph and backward wave speed -60 mph, with a capacity of 7200 vph and flow moving in the direction of increasing milepost. A work zone closure reduces the capacity to 3600 vph between mileposts 20 and 40 from 2:30 to 3:00, without affecting the free-flow speed, backward wave speed, or shape of the fundamental diagram. At milepost 0, the inflow rate is 1200 vph between 2:00 and 2:20; 4800 vph between 2:20 and 3:00; and 1200 vph between 3:00 and 3:20. At 2:00, the density is 40 veh/mi between mileposts 0 and 40, and 180 veh/mi between milepost 40 and 60. Report the average volumes at milepost 30 between 2:00 and 3:30, measured in 20-minute increments (that is, the average volume between 2:00 and 2:20, between 2:20 and 2:40, etc.). Report the average densities at 3:20 between mileposts 0 and 60, measured in 20-mile increments (that is, the average density between mileposts 0 and 20, 20 and 40, etc.)

From the given data, the jam density is 2(7200 veh/hr)/(60 mi/hr) = 240 veh/mi for typical flow and 2(3600 veh/hr)/(60 mi/hr) = 240 veh/mi. Outputs are required at 20-minute time intervals and 20 mile spatial intervals; we adopt this as the "mesh" for Daganzo's method. Forward and backward wave speeds are identical, and all links cover a timespan of 10 minutes. (Figure 2). The cost on all forward links is zero, while the cost on backward links is $k_j w \Delta t$ or (240 veh/mi)(60 mi/hr)(10 min) = 2400 veh for typical flow and (120 veh/mi)(60 mi/hr)(10 min) = 1200 veh in the work zone (lightly shaded region in Figure 2). Link costs are marked on the figure (all values given in hundreds).

Starting numbering with vehicle 0 at milepost 60 and 2 PM, the given density and volume values correspond to the cumulative counts shown along the left and bottom axes in the diagram (boldfaced and underlined). Cumulative counts within the diagram are underlined in plain text, based on shortest paths from a boundary point. Based on the results from Daganzo's method, the average density at 3:20 is 20 veh/mi between milepost 0 and 20; 100 veh/mi between 20 and 40; and 60 veh/mi between 40 and 60. The average flow at milepost 60 is 7200 veh/hr between 2:00 and 2:20; 6000 veh/hr between 2:20 and 2:40; 2400 veh/hr between 2:40 and 3:00; and 1200 veh/hr between 3:00 and 3:20.

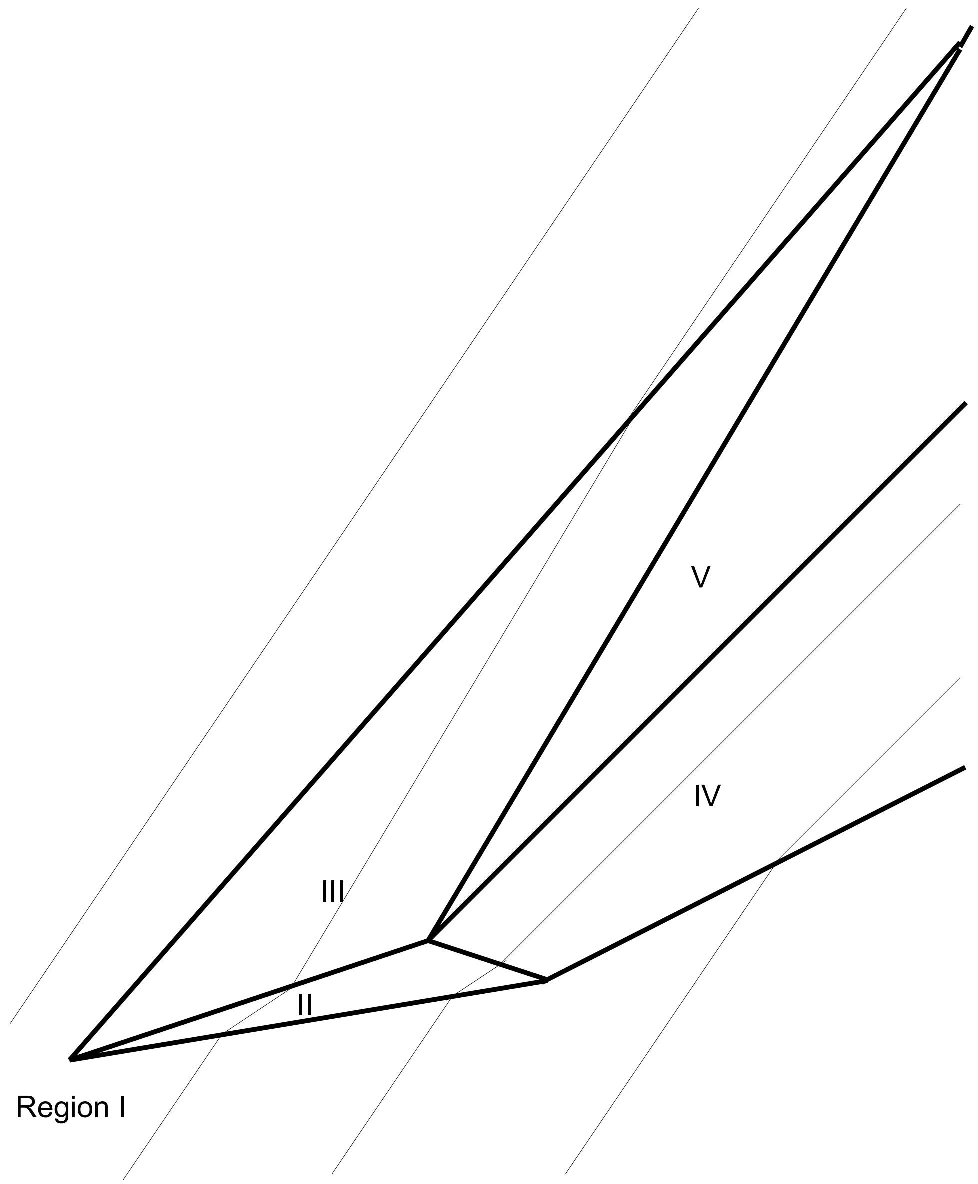

Figure 1: Shockwave diagram for Problem 2, with selected trajectories.

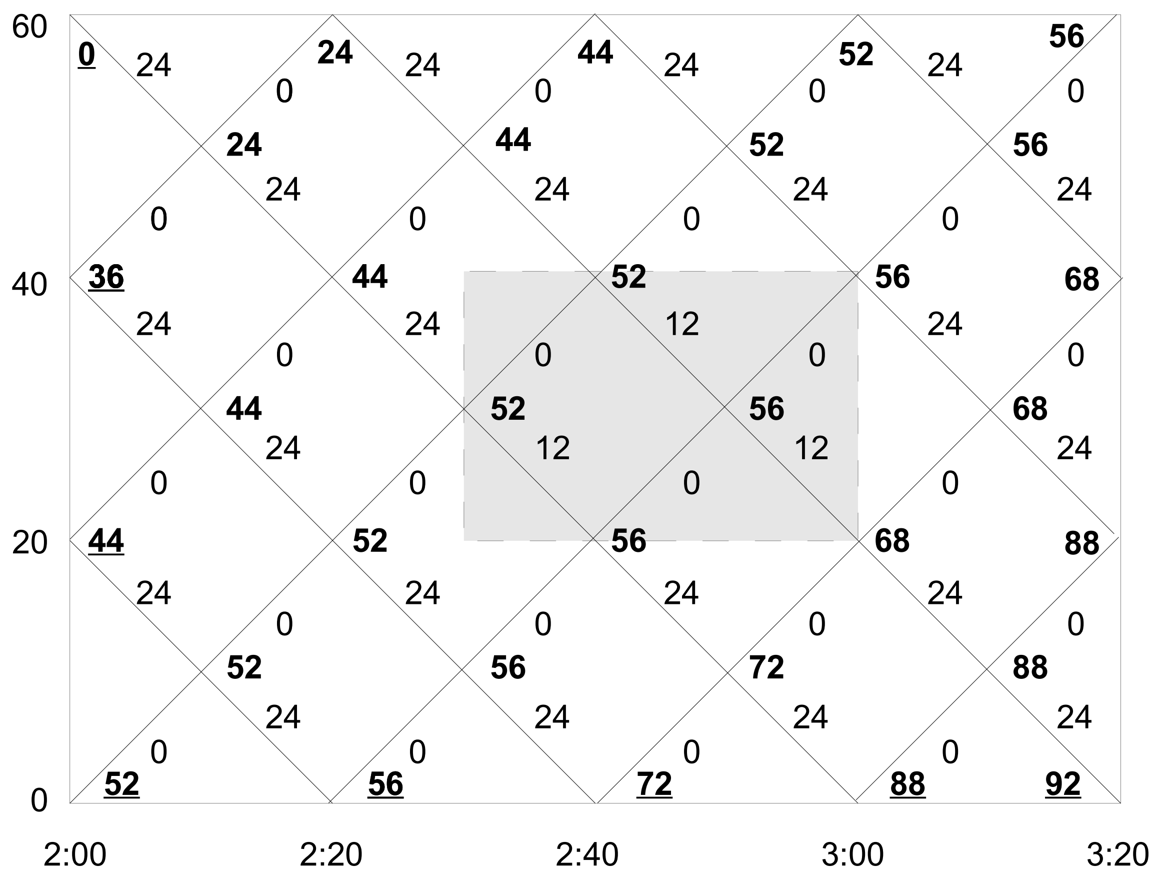

Figure 2: Daganzo's method for Problem 3. Boundary conditions underlined; all values in hundreds.