Dealing with path flow nonuniqueness: Entropy and proportionality

CE 392C

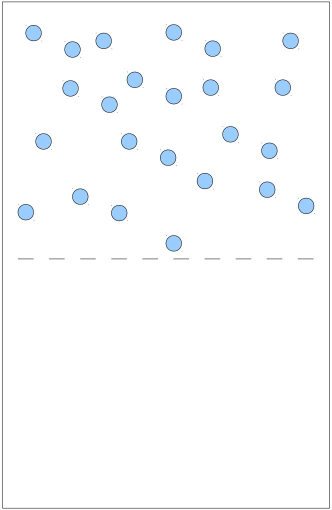
Uniqueness of equilibrium solutions

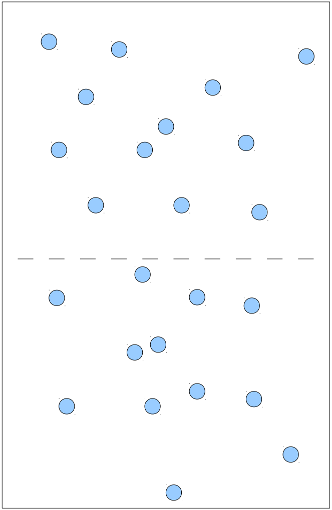
1. What's the problem?
2. Why do we care?

MOST LIKELY PATH FLOWS

The equilibrium principle is not strong enough to determine path flows in a network. So it is no use to ask “what is the equilibrium path flow solution?”

However, we can try to differentiate among the path flow solutions in another way, e.g. “What is the *most likely* equilibrium ptah flow solution?”





Why is the left situation far less likely than the right situation?

$$p^L = (1/2)^n$$

$$p^R = \binom{n}{n/2}\left(\frac{1}{2}\right)^{n/2}\left(\frac{1}{2}\right)^{n/2} = \frac{n!}{(n/2)!(n/2)!}\left(\frac{1}{2}\right)^n$$

Since $n! \gg [(n/2)!]^2$, the situation on the right is far more likely.

Under the assumption that drivers perceive routes identically and choose routes independently, the same logic holds for path flows!

Assume there are k equal travel time paths for a single OD pair, with constant travel time and integral demand d. Then

$$p = \frac{d!}{h_1! h_2! \cdots h_k!} \left(\frac{1}{k}\right)^d$$

and the most likely path flow vector maximizes

$$p = \frac{d!}{h_1! h_2! \cdots h_k!}$$

We might as well maximize

$$\log p = \log d! - \sum_{\pi=1}^{k} \log h_\pi!$$

Using Stirling's approximation this simplifies to

$$\log p \approx (d \log d - d) - \sum_{\pi} (h_\pi \log h_\pi - h_\pi)$$

or

$$\log p \approx - \sum_{\pi} h_\pi \log(h_\pi / d)$$

Changes for the general traffic assignment problem:

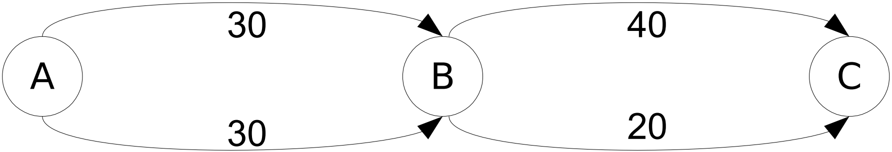
- Demand doesn't need to be an integer. No problem, actually helps us out by making Stirling's approximation exact.
- Multiple OD pairs. No problem, just multiply the probabilities for each OD pair (sum their log-probabilities).
- Travel times are flow dependent. Not a big problem, just add a constraint that the path flows correspond to equilibrium link flows.

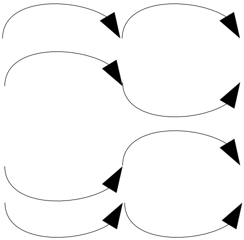
This is the *entropy maximization problem*:

$$\begin{aligned} \max_{\mathbf{h}} \quad & -\sum_{(r,s)\in Z^2} \sum_{\pi\in\hat{\Pi}^{rs}} h_\pi \log(h_\pi/d^{rs}) & \\ \text{s.t.} \quad & \sum_{\pi\in\Pi} \delta_{ij}^\pi h_\pi = x_{ij}^* & \forall (i,j) \in A \\ & \sum_{\pi\in\hat{\Pi}^{rs}} h_\pi = d^{rs} & \forall (r,s) \in Z^2 \\ & h_\pi \geq 0 & \forall \pi \in \Pi \end{aligned}$$

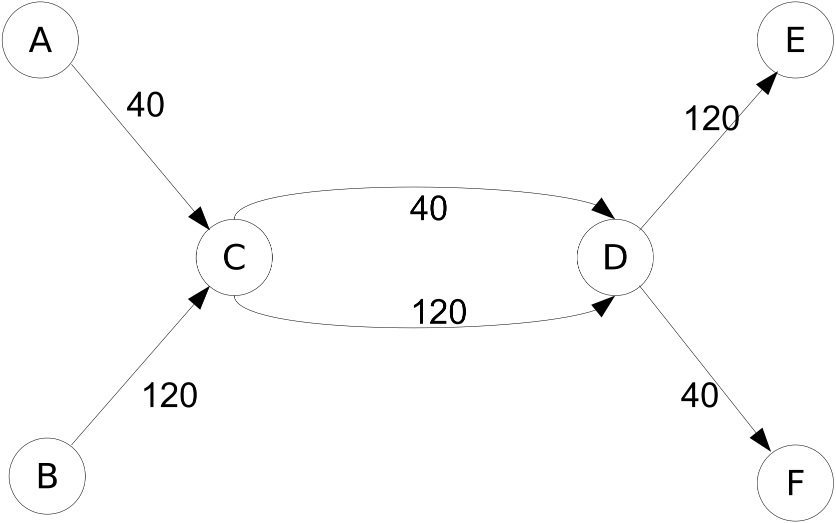
Where does the term entropy come from?

ENTROPY MAXIMIZATION AND PROPORTIONALITY





Path	#	h_1	h_2
	1	20	30
	2	10	0
	3	20	10
	1	10	20


A
E
40
120
40
C
D
120
120
40
B
F

PROPORTIONAL LINK FLOWS

This is the *entropy maximization problem*:

$$\begin{aligned} \max_{\mathbf{h}} \quad & -\sum_{(r,s)\in Z^2} \sum_{\pi \in \hat{\Pi}^{rs}} h_\pi \log(h_\pi / d^{rs}) & \\ \text{s.t.} \quad & \sum_{\pi \in \Pi} \delta_{ij}^\pi h_\pi = x_{ij}^* & \forall (i,j) \in A \\ & \sum_{\pi \in \hat{\Pi}^{rs}} h_\pi = d^{rs} & \forall (r,s) \in Z^2 \\ & h_\pi \geq 0 & \forall \pi \in \Pi \end{aligned}$$

A more formal way to state the proportionality condition is as follows:

A link flow vector $\mathbf{h}$ satisfies proportionality if, for every paths π_1 and π_2 connecting the same OD pair, the ratio h_1/h_2 only depends on the pair(s) of alternate segments distinguishing these paths. (In particular, it doesn't depend on which OD pair π_1 and π_2 connect.)

Proportionality can be derived from the optimality conditions for this problem. The Lagrangian is

$$\mathcal{L}(\mathbf{h}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = -\sum_{(r,s)\in Z^2} \sum_{\pi \in \hat{\Pi}^{rs}} h_\pi \log\left(\frac{h_\pi}{d^{rs}}\right) + \sum_{(i,j)\in A} \beta_{ij}\left(x^*_{ij} - \sum_{\pi\in\Pi} \delta^\pi_{ij} h_\pi\right) + \sum_{(r,s)\in Z^2} \gamma_{rs}\left(d^{rs} - \sum_{\pi\in\hat{\Pi}^{rs}} h_\pi\right)$$

The nonnegativity constraint $h_\pi \geq 0$ is redundant and can be ignored (why?), so the only optimality condition is that $\nabla\mathcal{L} = \mathbf{0}$

From $\frac{\partial \mathcal{L}}{\partial h^\pi} = 0$ we have

$$h^\pi = d^{rs} \exp\left(-1 - \sum_{(i,j)\in A} \delta_{ij}^\pi \beta_{ij} - \gamma_{rs}\right)$$

From $\frac{\partial \mathcal{L}}{\partial \gamma_{rs}} = 0$ we have

$$d^{rs} = \sum_{\pi' \in \hat{\Pi}^{rs}} h_{\pi'} = d^{rs} \exp\left(-1 - \gamma_{rs}\right) \sum_{\pi \in \hat{\Pi}^{rs}} \exp\left(-\sum_{(i,j)\in A} \delta_{ij}^\pi \beta_{ij}\right)$$

so

$$\gamma_{rs} = -1 + \log\left(\sum_{\pi' \in \hat{\Pi}^{rs}} \exp\left(-\sum_{(i,j)\in A} \delta_{ij}^{\pi'} \beta_{ij}\right)\right)$$

Therefore

$$h^{\pi} = \frac{\gamma_{rs}}{\sum_{\pi' \in \hat{\Pi}^{rs}} \exp\left(-\sum_{(i,j)\in A} \delta_{ij}^{\pi'} \beta_{ij}\right)} \exp\left(-\sum_{(i,j)\in A} \delta_{ij}^{\pi} \beta_{ij}\right)$$

However, the fraction is a constant only depending on the OD pair (r,s), so we can write

$$h^{\pi} = K_{rs} \sum_{(i,j)\in A} \exp\left(-\delta_{ij}^{\pi} \beta_{ij}\right)$$

Therefore, for any two paths π_1 and π_2 connecting the same OD pair, we have

$$\frac{h_1}{h_2} = \frac{K_{rs} \exp\left(-\sum_{(i,j)\in A} \delta_{ij}^{\pi_1} \beta_{ij}\right)}{K_{rs} \exp\left(-\sum_{(i,j)\in A} \delta_{ij}^{\pi_2} \beta_{ij}\right)}$$

In gradient projection, we partitioned the arcs into four sets: A_1, A_2, A_3, and A_4 for links which were on neither path, on both paths, just on π_1, and just on π_2.

Doing the same here and cancelling common factors, we have

$$\frac{h_1}{h_2} = \frac{\exp\left(-\sum_{(i,j)\in A_3} \delta_{ij}^{\pi_1} \beta_{ij}\right)}{\exp\left(-\sum_{(i,j)\in A_4} \delta_{ij}^{\pi_1} \beta_{ij}\right)}$$

This ratio only depends on the the pairs of alternate segments distinguishing the paths (A_3 and A_4), and NOT on the OD pair.

Therefore, *entropy-maximizing path flows satisfy the proportionality condition.*

Unfortunately, the reverse isn't true, and you can create examples where proportionality is satisfied without maximizing entropy.

However, in practical terms proportionality "gets you most of the way there" and is much easier to obtain and verify.

In the Chicago regional network, there are roughly 93 million equilibrium paths. (So 93 million degrees of freedom in choosing path flows.)

Accounting for the equilibrium and demand constraints, there are still 90 million degrees of freedom.

After enforcing proportionality, there are only 91 dof left, a reduction of 99.9999%!

HOW TO FIND HIGH-ENTROPY SOLUTIONS

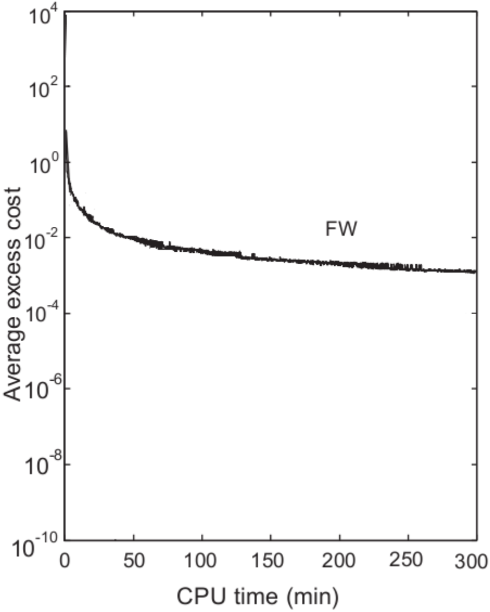
The proportionality condition implies that high-entropy solutions *spread flow over as many paths as possible.*

This implies that path-based algorithms *generally produce low-entropy solutions*. Why?

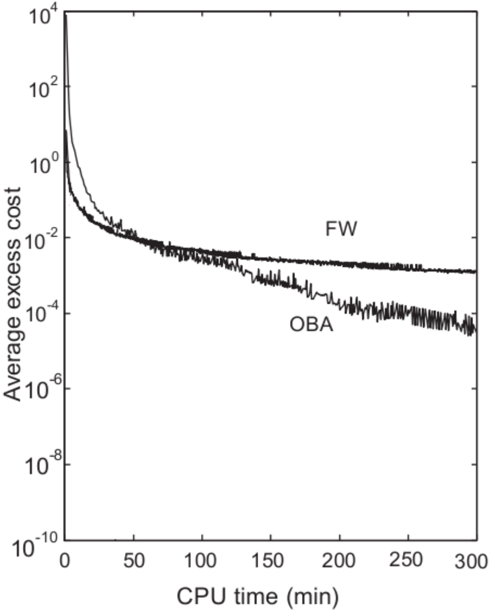
By contrast, bush-based algorithms can produce higher-entropy solutions without running into memory problems.

TAPAS is a bush-based algorithm which aims to simultaneously find equilibrium and achieve proportionality.

Empirically, it performs very well.



(Adapted from Bar-Gera, 2010)



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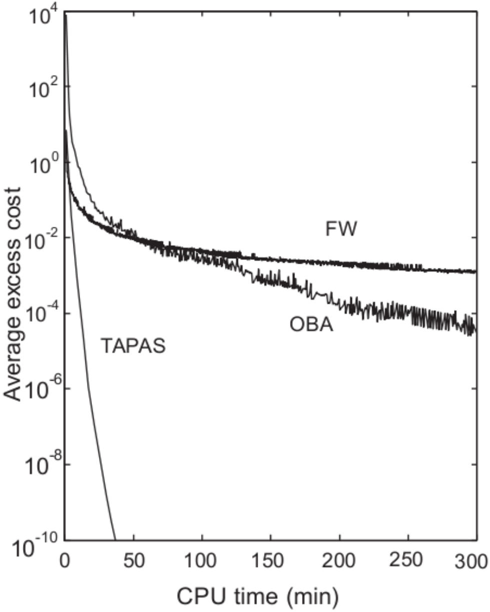

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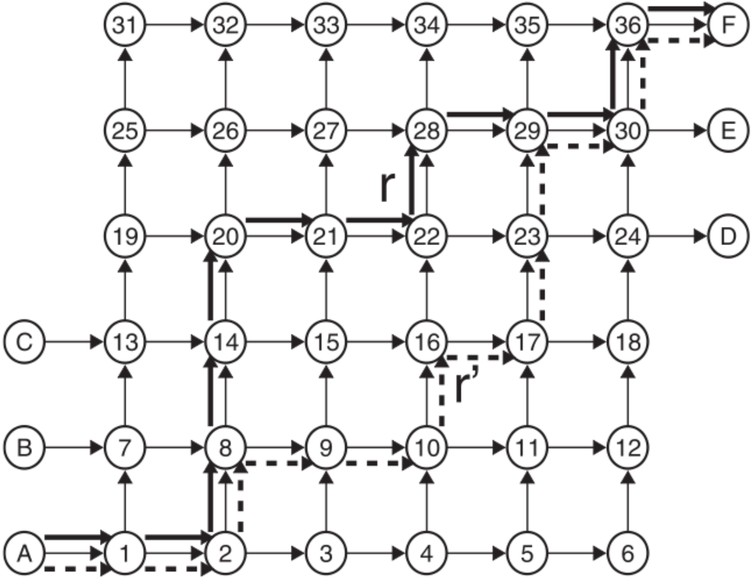
Traffic Assignment by Paired Alternative Segments

TAPAS is a bush-based algorithm which also tracks a set of *paired alternative segments*



(Bar-Gera, 2010)

Traffic Assignment by Paired Alternative Segments

TAPAS is a bush-based algorithm which also tracks a set of *paired alternative segments* (PASs)

The idea is that algorithms like gradient projection or B will shift flow between the same sets of links over and over, and that it may be more efficient to store them.

Furthermore, each PAS may be used by multiple origins. By tracking this, we can help ensure proportionality is satisfied.

TAPAS involves two main steps: *flow shifts*, and *proportionality adjustments*.

Here is an example of a PAS between nodes 42 and 88:

Segment 1:

Link	Time	Origin 11	Origin 73	Origin 102	Origin 137	Total
(42,23)	5	4	9	4	15	32
(23,45)	3	2	4	7	32	45
(45,88)	12	7	6	2	8	16

Segment 2:

Link	Time	Origin 11	Origin 73	Origin 102	Origin 137	Total
(42,103)	5	7	3	7	8	25
(103,57)	2	5	2	13	6	26
(57,58)	3	5	4	0	9	18
(58,88)	6	12	1	23	5	41

Segment 1 has a higher travel time than Segment 2, so we shift flow from Segment 1 to Segment 2

Flow shifts

The question of *how much flow* to shift is answered the same way as before (Newton's method). The new question is how much flow to shift *from each origin*.

Let's say that Newton's method tells us to shift 4 vehicles from Segment 1 to Segment 2.

First, we need to figure out how many vehicles from each origin are actually using all of Segment 1; this tells us the maximum amount we can shift away from Segment 1.

Segment 1:

Link	Time	Origin 11	Origin 73	Origin 102	Origin 137	Total
(42,23)	5	4	9	**4**	15	32
(23,45)	3	**2**	4	7	32	45
(45,88)	12	7	6	**2**	**8**	16
Minimum		2	4	2	8	16

Only 16 vehicles are using all of Segment 1; the other flow consists of vehicles entering or leaving midway through.

Flow shifts

The question of *how much flow* to shift is answered the same way as before (Newton's method). The new question is how much flow to shift *from each origin*.

Let's say that Newton's method tells us to shift 4 vehicles from Segment 1 to Segment 2.

First, we need to figure out how many vehicles from each origin are actually using all of Segment 1; this tells us the maximum amount we can shift away from Segment 1.

So we can shift at most 16 vehicles. $4 \leq 16$, so we don't need to truncate the shift. The number of vehicles to shift from each origin is *proportional to the minimum values found in the previous step.*

Segment 1:

Link	Time	Origin 11	Origin 73	Origin 102	Origin 137	Total
(42,23)	5	4	9	**4**	15	32
(23,45)	3	**2**	4	7	32	45
(45,88)	12	7	6	**2**	**8**	16
Minimum		2	4	2	8	16
Shift		0.5	1	0.5	2	4

So, after the shift we have:

Segment 1:

Link	Origin 11	Origin 73	Origin 102	Origin 137	Total
(42,23)	3.5	8	3.5	13	28
(23,45)	1.5	3	6.5	30	41
(45,88)	6.5	5	1.5	6	12

Segment 2:

Link	Origin 11	Origin 73	Origin 102	Origin 137	Total
(42,103)	7.5	4	7.5	10	25
(103,57)	5.5	3	13.5	8	26
(57,58)	5.5	5	0.5	11	18
(58,88)	12.5	2	23.5	7	41

Proportionality adjustment

In a proportionality adjustment, we make sure that a particular PAS satisfies the proportionality condition. This is easy on an "isolated" PAS (without traffic entering or leaving midway through)

Origin	Segment 1 flow	Segment 2 flow	Proportion
3	15	10	3/5
25	15	0	1
43	50	10	5/6
Total	80	20	4/5

We shift the flows between segments in such a way that *the total flow on each segment is unaffected, but proportionality is achieved.*

Origin	Flow 1	Flow 2	Proportion	Shift	New flow 1	New flow 2
3	15	10	$3/5$	$+5$	20	5
25	15	0	1	-3	12	3
43	50	10	$5/6$	-2	48	12
Total	80	20	$4/5$	0	80	20

Notice that the total flow (and thus travel time) on each segment is the same, but the proportionality condition is now satisfied.

(On a non-isolated PAS, this process is more difficult; refer to Bar-Gera's paper for details.)

The overall TAPAS framework is as follows:

1. Find initial solution
2. For each origin:
    1. Update bush
    2. Identify PASs which provide shortcuts (based on links outside the bush)
    3. Shift flow within PASs
3. For every PAS:
    1. Perform flow shift
    2. Perform proportionality adjustment
    3. Delete if no longer used
4. Return to step 2 unless gap is sufficiently small.
5. Perform additional proportionality adjustments.