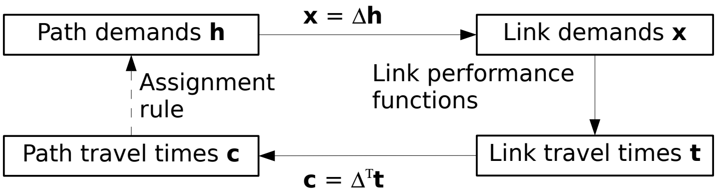
User equilibrium and system optimum

CE 392C

OUTLINE

1. Traffic assignment on simple networks
2. The notion of equilibrium
3. Some motivating examples

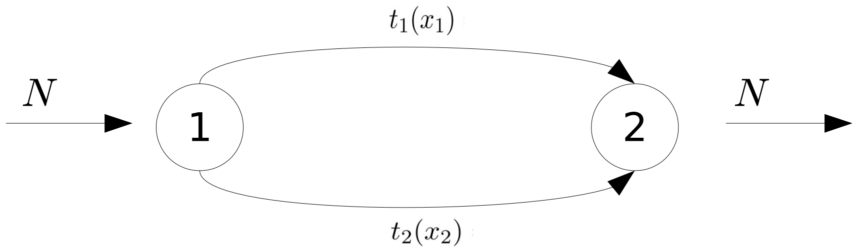
We left one arrow undefined, the "assignment rule." Dealing with this will require the rest of the semester.



Why do travelers choose the paths they do?

USER EQUILIBRIUM

Assume that there are only two network links, and N vehicles traveling from the origin to the destination.



If all drivers are choosing routes that minimize their delay, then exactly one of the following cases must be true:

1. All drivers are on the top link, and $t_1(N) < t_2(0)$.
2. All drivers are on the bottom link, and $t_2(N) < t_1(0)$.
3. The two links have equal travel time $t_1(x_1) = t_2(x_2)$

Anything else is “unstable” and some drivers will switch routes.

This can be generalized in the **principle of user equilibrium**:

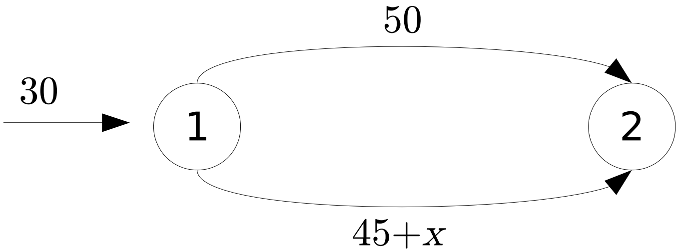
For each origin and destination, all used routes between those nodes have equal and minimal travel time.

Note that this principle follows directly from the assumptions that drivers choose minimum time paths, and are well-informed about network conditions.

If you accept these assumptions, then you must also accept the principle of user equilibrium.

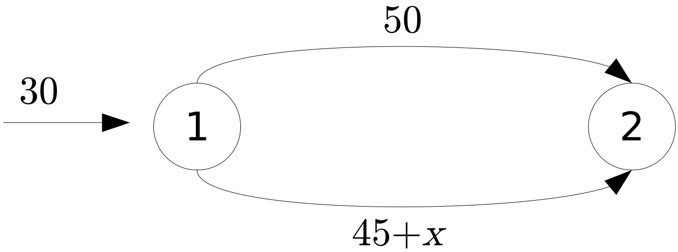
Equivalently, if you disagree with the principle of user equilibrium, then you must either believe that drivers do not choose minimum time paths, or do not know the travel times on available paths.

We can use this principle to solve for equilibrium on very simple networks. If there are 30 vehicles choosing these routes, how many choose the top route, and how many choose the bottom?

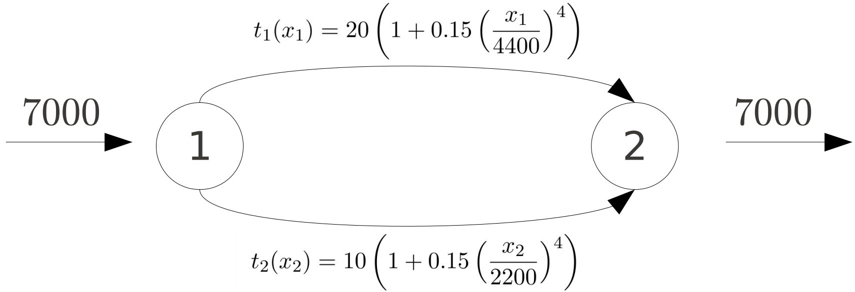


What do these link performance functions imply?

25 choose the top route, 5 choose the bottom, and everybody has a travel time of 50 minutes.

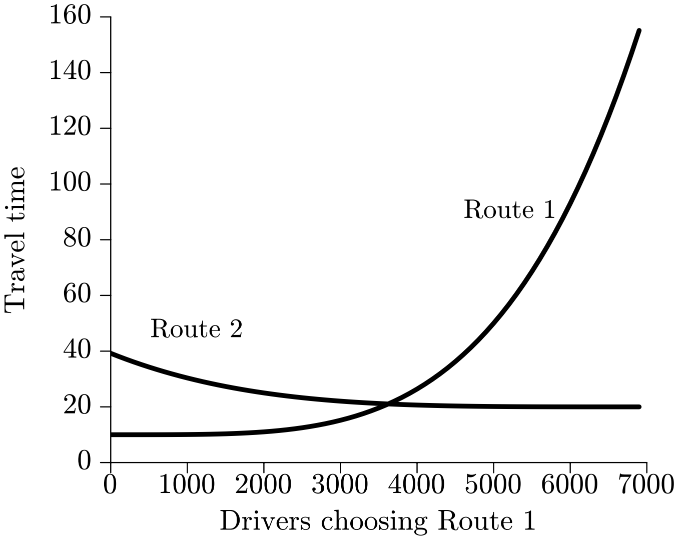


Even with more complex functions we can apply a similar approach.



An equation solver may be needed in this case.

A graphical method can be used when there are only 2 paths.

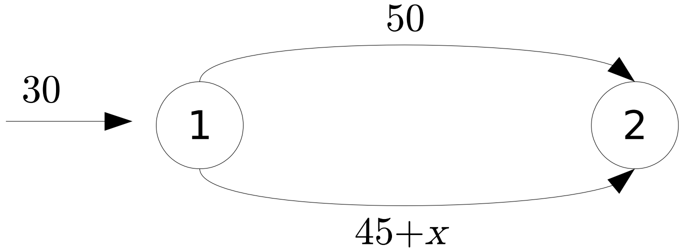


This method can be generalized in any network with a single OD pair (r, s):

1. Select a set of paths $\hat{\Pi}^{rs}$ which you think will be used.
2. Write equations for the travel times of each path in $\hat{\Pi}^{rs}$ as a function of the path demands.
3. Solve the system of equations enforcing equal travel times on all of these paths, together with the requirement that the total path demands must equal the total demand d^{rs}.
4. Verify that this set of paths is correct; if not, refine $\hat{\Pi}^{rs}$ and return to step 2.

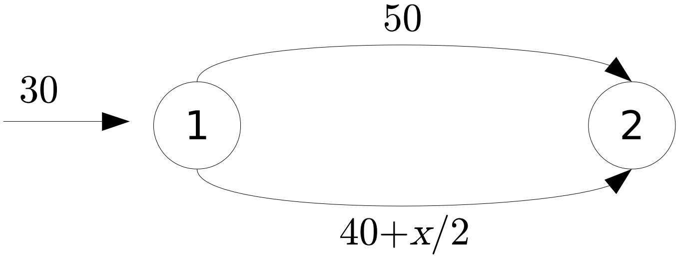
KNIGHT-PIGOU-DOWNS PARADOX

Remember this network from the first example?



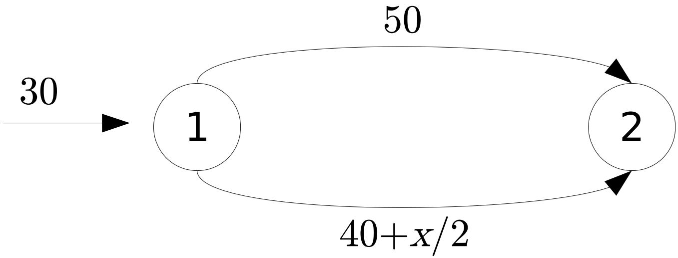
At equilibrium, 25 vehicles chose the top route, 5 chose the bottom, all travel times are 50 minutes.

Now, we **improve** the bottom link so that its cost function is now $40 + x/2$. What happens to route choices now?



Why is changing the cost function in this way an "improvement"?

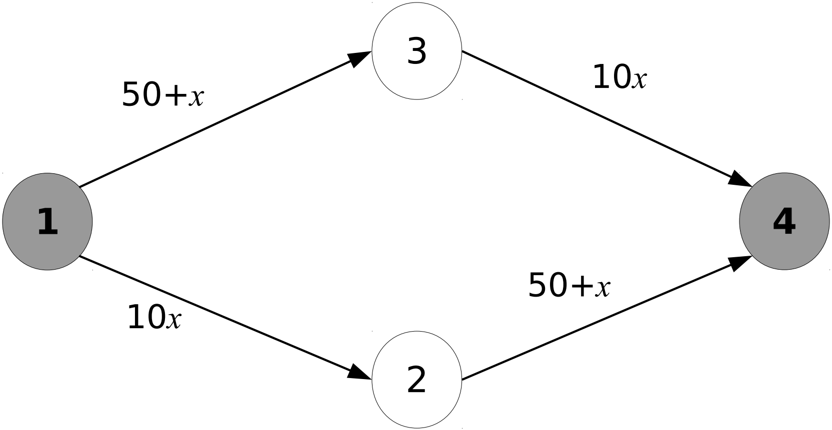
10 vehicles choose the top route, 20 choose the bottom route, and everybody has a travel time of 50 minutes.



Nobody has saved any time at all! What happened?

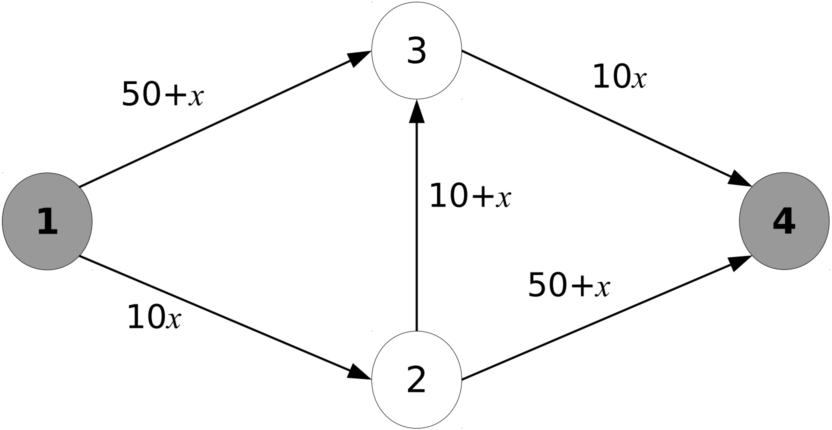
BRAESS PARADOX

Consider the following network, with 6 vehicles traveling from node 1 to node 4



What's the equilibrium solution?

Now, a third link is added to the network.



What happens now?

What just happened?

Is the Braess paradox “realistic”?

A few implications:

- User equilibrium does *not* minimize congestion.
- The “invisible hand” does not always function well in traffic networks.
- There may be room for engineers and policy makers to “improve” route choices.

This suggests two possible traffic assignment rules:

User equilibrium (UE): Find a feasible assignment in which all used paths have equal and minimal travel times.

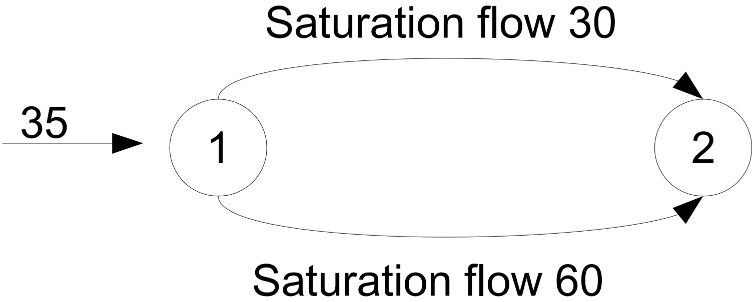
System optimum (SO): Find a feasible assignment which minimizes the total system travel time

$$TSTT = \sum_{(i,j) \in A} x_{ij} t_{ij}$$

When might each of these rules be used?

SMITH’S PARADOX

Assume there are two routes which merge at a signal-controlled junction, with a 60-second cycle length.



The saturation flow is the maximum throughput on a link if it were given 100% green time.

From traffic engineering principles, we can estimate the signal delay on each link in terms of its flow, and the percentage of green time:

$$t_i = 1 + \frac{9}{20}\left[\frac{C(1 - G_i/C)^2}{1 - x_i/s_i} + \frac{X_i^2}{x_i(1 - X_i)}\right]$$

(To focus on the main issues, I will just present the results of the calculations in this example, rather than going through this formula each time.)

Initially, the signal is timed so the top approach has 40 seconds of green, and the bottom has 20 seconds.

With these values, the equilibrium solution is $x^{\uparrow} = 23.6$ and $x^{\downarrow} = 11.4$; both approaches have a travel time of 2.11 minutes.

With these flow values, delay at the signal is minimized by adjusting the green times to 48.3 and 11.7 seconds.

This changes the equilibrium solution. With the new delay functions, the equilibrium is $x^{\uparrow} = 23.8$ and $x^{\downarrow} = 11.2$; both approaches have a travel time of 2.26 minutes.

With these flow values, delay at the signal is minimized by adjusting the green times to 48.5 and 11.5 seconds.

As this process continues, delay continues to increase... at first slowly, and then dramatically:

Iteration	$G^{\uparrow}$ (s)	$G^{\downarrow}$	$x^{\uparrow}$ (v/min)	$x^{\downarrow}$	$t^{\uparrow}$ (min)	$t^{\downarrow}$
0	48	12	23.6	11.4	2.11	2.11
1	48.3	11.7	23.8	11.2	2.26	2.26
2	48.5	11.5	23.9	11.1	2.43	2.43
3	48.7	11.3	24.1	10.9	2.63	2.63
4	48.9	11.1	24.2	10.8	2.86	2.86
5	49.1	10.9	24.3	10.7	3.12	3.12
10	49.5	10.5	24.7	10.3	5.11	5.11
20	49.88	10.12	24.91	10.09	16.58	16.58
50	49.998	10.002	24.998	10.002	855.92	855.93
∞	50	10	25	10	∞	∞