Equilibrium with elastic demand

CE 392C

In the next few weeks, we'll look at variations of the basic problem to try and make it more realistic.

The specific variations are

Elastic demand : The OD matrix should depend on congestion, rather than be fixed.

Destination choice : Another twist on elastic demand: the destinations drivers choose depend partly on congestion.

Perception error : Drivers don't know travel times exactly, only approximately.

There are many others, but this gives you a good introduction to the flavor of such extensions.

This is the way research generally works: start with a simplified version of the problem; once you figure out how to solve that, gradually make it more realistic.

ELASTIC DEMAND

To date, we've assumed that the demand for travel between origin r and destination s is a fixed value d^{rs}.

Now, assume instead that d^{rs} is given by a *demand function depending on the travel time from r to s*:

$$d^{rs} = D^{rs}(\kappa^{rs})$$

What kind of function should D^{rs} be?

D^{rs} should probably be nonincreasing, nonnegative, and bounded over possible values of κ^{rs}.

Furthermore it will be very convenient to assume that it is invertible, that is, that the inverse function D_{rs}^{-1} exists and is well defined: D_{rs}^{-1} gives us the *travel time* between r and s for a given demand level.

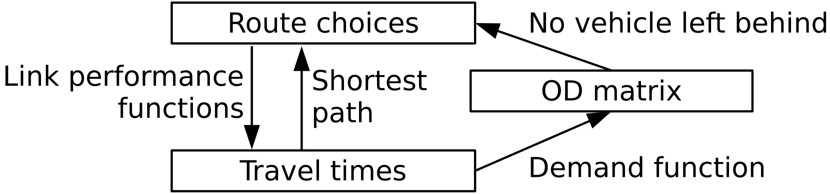
This immediately causes a problem — why?

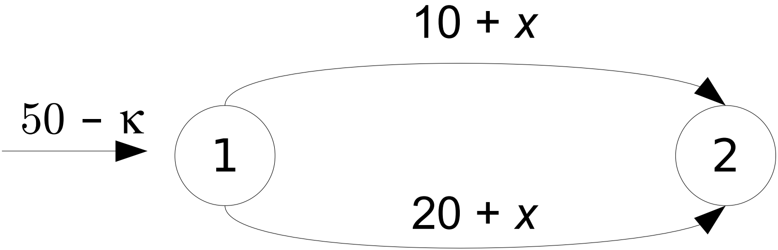
It turns out that invertibility is more important than nonnegativity: we will just add a condition that says that if D^{rs} is negative, we will make the demand zero: $d^{rs} = [D^{rs}(\kappa^{rs})]^{+}$.

Conceptually, in the basic model $\mathbf{d}$ is exogenous, and we determined $\mathbf{x}$ endogenously.

With elastic demand, both $\mathbf{d}$ and $\mathbf{x}$ are endogenous and can affect each other.

We've added another "consistency" relationship:




10 + x
50 − κ
1
2
20 + x

We can solve this algebraically. Assume both paths are used. Then:

$$\begin{aligned} d &= 50 - \kappa \\ x_1 + x_2 &= d \\ 10 + x_1 &= 20 + x_2 \\ \kappa &= 10 + x_1 \end{aligned}$$

and the solution to this system of equations is $x_1 = 16\frac{2}{3}$, $x_2 = 6\frac{2}{3}$, $d = 23\frac{1}{3}$, $\kappa = 26\frac{2}{3}$.

Of course, we'll need another approach for larger networks.

As an example application, what would happen if the bottom link were "upgraded" so its delay function is $10 + x_2$ instead of $20 + x_2$?

If the demand were **fixed** at $23\frac{1}{3}$, the new solution would be $x_1 = x_2 = 11\frac{2}{3}$, and the equilibrium travel time would be $21\frac{2}{3}$ (a reduction of 5 minutes).

However, if we re-solve the elastic demand formulation, the new solution is $x_1 = x_2 = 13\frac{1}{3}$, and the equilibrium travel time is $23\frac{1}{3}$ (a reduction of only 3 minutes). Why?

Did the previous demand function satisfy the conditions for a “reasonable” demand function?

OPTIMIZATION FORMULATION

Just as an optimization formulation lets us solve the basic equilibrium problem on large networks, the same technique can be applied for the elastic demand equilibrium problem.

As a reminder, the basic equilibrium problem solved

$$\begin{aligned}
\min_{\mathbf{x},\mathbf{h}} \quad & \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x)dx & \\
\text{s.t.} \quad & x_{ij} = \sum_{\pi\in\Pi} \delta_{ij}^{\pi} h^{\pi} & \forall (i,j) \in A \\
& d^{rs} = \sum_{\pi\in\Pi^{rs}} h^{\pi} & \forall (r,s) \in Z^2 \\
& h^{\pi} \geq 0 & \forall \pi \in \Pi
\end{aligned}$$

because its optimality conditions are

$$\begin{aligned}
h^{\pi} &\geq 0 & \forall \pi \in \Pi \\
c^{\pi} &\geq \kappa_{rs} & \forall (r,s) \in Z^2 \\
h^{\pi}(c^{\pi} - \kappa_{rs}) &= 0 & \forall \pi \in \Pi \\
\sum_{\pi\in\Pi^{rs}} h^{\pi} &= d^{rs} & \forall (r,s) \in Z^2
\end{aligned}$$

If we make d_{rs} a decision variable, what new conditions do we need for elastic demand?

if $d_{rs} > 0$, then it equals the demand function; if $d_{rs} = 0$, then the demand function must be nonpositive.

This sounds suspiciously similar to the equilibrium principle: if $h^\pi > 0$, then its travel time equals κ_{rs}; if $h^\pi = 0$, then $C^\pi \geq \kappa_{rs}$

In fact, the following conditions express this relationship:

$$\begin{aligned} d_{rs} &\geq 0 & \forall (r,s) \in Z^2 \\ d_{rs} &\geq D_{rs}(\kappa_{rs}) & \forall (r,s) \in Z^2 \\ d_{rs}(d_{rs} - D_{rs}(\kappa_{rs})) &= 0 & \forall (r,s) \in Z^2 \end{aligned}$$

It turns out to be easier to rewrite the last two conditions in terms of the inverse demand function: if $d_{rs} = D_{rs}(\kappa_{rs})$, then $\kappa_{rs} = D_{rs}^{-1}(d_{rs})$.

The demand function D_{rs} tells us how many trips will be made for a given level of congestion (travel time). The inverse demand function D_{rs}^{-1} tells us what the travel time must be if we know the total number of trips made.

In this way, the new conditions become:

$$\begin{aligned} d_{rs} &\geq 0 && \forall (r,s) \in Z^2 \\ \kappa_{rs} &\geq D_{rs}^{-1}(d_{rs}) && \forall (r,s) \in Z^2 \\ d_{rs}(\kappa_{rs} - D_{rs}^{-1}(d_{rs})) &= 0 && \forall (r,s) \in Z^2 \end{aligned}$$

We need to write the Lagrangian function $\mathcal{L}(\mathbf{h}, \mathbf{d}, \boldsymbol{\kappa})$ so that the following expressions replicate these conditions (plus the equilibrium ones):

$$h^\pi \geq 0 \qquad \forall \pi \in \Pi \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial h^\pi} \geq 0 \qquad \forall \pi \in \Pi \tag{2}$$

$$h^\pi \frac{\partial \mathcal{L}}{\partial h^\pi} = 0 \qquad \forall \pi \in \Pi \tag{3}$$

$$d_{rs} \geq 0 \qquad \forall (r,s) \in Z^2 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial d_{rs}} \geq 0 \qquad \forall (r,s) \in Z^2 \tag{5}$$

$$d_{rs} \frac{\partial \mathcal{L}}{\partial d_{rs}} = 0 \qquad \forall (r,s) \in Z^2 \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \kappa_{rs}} = 0 \qquad \forall (r,s) \in Z^2 \tag{7}$$

If we start with $\mathcal{L} = \sum_{ij} \int_0^{\sum_\pi \delta_{ij}^\pi h^\pi} t_{ij}(x)\ dx + \sum_{rs} \left(d_{rs} - \sum_{\pi \in \Pi_{rs}} h^\pi\right)$ then conditions (1), (2), (3), and (7) still satisfy the same conditions as in the basic equilibrium problem. (4) corresponds to the first elastic demand condition.

If we can get $\frac{\partial \mathcal{L}}{\partial d_{rs}}$ to equal $\kappa_{rs} - D_{rs}^{-1}(d_{rs})$, then we're done.

Note that the partial derivative of our "starter Lagrangian" wrt d_{rs} is κ_{rs}. So, all we need to do is subtact the term $\int_0^{d_{rs}} D_{rs}^{-1}(\omega)\ d\omega$.

That is, the Lagrangian should be

$$\sum_{ij} \int_0^{\sum_\pi \delta_{ij}^\pi h^\pi} t_{ij}(x)\ dx - \sum_{rs} \int_0^{d_{rs}} D_{rs}^{-1}(\omega)\ d\omega + \sum_{rs} \left(d_{rs} - \sum_{\pi \in \Pi_{rs}} h^\pi\right)$$

In other words, our optimization formulation can be

$$
\begin{aligned}
\min_{\mathbf{x},\mathbf{h},\mathbf{d}} \quad & f(\mathbf{x},\mathbf{d}) = \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x)\, dx \quad - \sum_{(r,s)\in Z^2} \int_0^{d_{rs}} D_{rs}^{-1}(\omega)\, d\omega \\
\text{s.t.} \quad & x_{ij} = \sum_{\pi\in\Pi} \delta_{ij}^{\pi} h^{\pi} && \forall (i,j) \in A \\
& d_{rs} = \sum_{\pi\in\Pi^{rs}} h^{\pi} && \forall (r,s) \in Z^2 \\
& h^{\pi} \geq 0 && \forall \pi \in \Pi \\
& d_{rs} \geq 0 && \forall \pi \in \Pi
\end{aligned}
$$

This is why using the inverse demand function is more convenient: both t and D^{-1} have the same units, so we can just add their integrals together.

SOLUTION METHOD

We can adapt both MSA and Frank-Wolfe to solve the elastic demand problem; only a few steps need to change:

- Instead of simply keeping track of current link flows $\mathbf{x}$, we also need to track the current OD matrix $\mathbf{d}$
- In addition to finding target link flows $\mathbf{x}^*$, we also need to find a target OD matrix $\mathbf{d}^*$
- For Frank-Wolfe, we need to find the derivative of the new objective function with respect to λ
- We need to expand the termination criteria to include both the equilibrium condition and the elastic demand condition.

The first step is fairly simple. Let's focus on the other two.

Here is a way to calculate the target solution $(\mathbf{d}^*, \mathbf{x}^*)$:

- For every OD pair, calculate the shortest path travel time κ_{rs}
- Let $d^*_{rs} = D_{rs}(\kappa_{rs})$ (calculate the target demand based on current link costs)
- Obtain $\mathbf{x}^*$ by loading the target demand $\mathbf{d}^*$ onto shortest paths.

With this choice of $\mathbf{d}^*$ and $\mathbf{x}^*$, writing the objective function in terms of λ, we can show that

$$\frac{df}{d\lambda} = \sum_{ij} t_{ij}(x'_{ij})(x^*_{ij} - x_{ij}) - \sum_{rs} D^{-1}_{rs}(d'_{rs})(d^*_{rs} - d_{rs})$$

where $\mathbf{x}' = \lambda \mathbf{x}^* + (1-\lambda)\mathbf{x}$ and $\mathbf{d}' = \lambda \mathbf{d}^* + (1-\lambda)\mathbf{d}$

You can find a zero of this function as before (either bisection or direct equation solving).

The next homework will ask you to derive this formula — extra credit if you can also show that $\mathbf{d}^* - \mathbf{d}$ is an improving direction, that is, $df/d\lambda \leq 0$ at $\lambda = 0$.

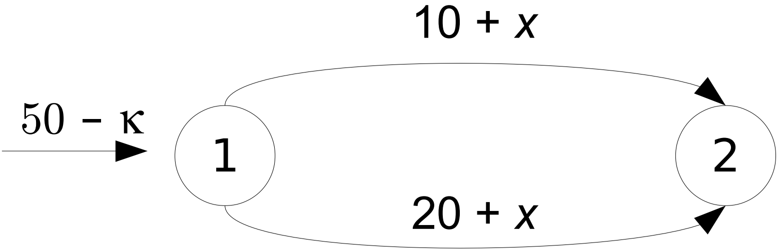
Termination criteria

For the equilibrium condition, we can use the same convergence criteria as before (average excess cost or relative gap).

For the demand condition, we can use the **total misplaced flow**:

$$TMF = \sum_{(r,s) \in Z^2} \left| d^{rs} - [D^{rs}(\kappa^{rs})]^+ \right|$$

At termination, **both** the average excess cost and total misplaced flow should be small.


10 + x
50 − κ
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Starting with the initial solution $\begin{bmatrix} d & x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 50 & 50 & 0 \end{bmatrix}$ we find the target $\begin{bmatrix} d^* & x_1^* & x_2^* \end{bmatrix} = \begin{bmatrix} 30 & 0 & 30 \end{bmatrix}$

Writing the expression for the derivative and simplifying, we obtain

$$(10 + 50(1 - \lambda))(-50) + (20 + 30\lambda)(30) - (50 - (30\lambda + 50(1 - \lambda)))(-20)$$

which vanishes if $\lambda = 12/19 \approx 0.632$

So the new solution is $\begin{bmatrix} d & x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 37.36 & 18.42 & 18.95 \end{bmatrix}$

GARTNER TRANSFORMATION

There is a clever way to solve the elastic demand problem as a regular equilibrium problem, involving a network transformation.

Add new links directly connecting every origin to every destination; let $\overline{d}_{rs}$ be an upper bound on the demand from r to s.

The cost function on each such link (r, s) is $D_{rs}^{-1}(\overline{d}_{rs} - x_{rs})$.

Let the fixed demand from r to s be $\overline{d}_{rs}$.

Then a solution to the basic equilibrium problem on the transformed network corresponds to an elastic demand equilibrium on the original network.