Larger worked-out example of Frank-Wolfe for elastic demand

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October 15, 2018

Section 9.1.6 of the text contained a worked-out example of Frank-Wolfe for elastic demand, using a network with two links. This document is a second worked-out example of the same algorithm, on a slightly larger network. (Figure 1). The figure shows the link performance functions; the only OD pair is from node 1 to node 4, and the demand function is

$$D^{14}(\kappa^{14}) = 15 - \kappa^{14}/10\,.$$

The corresponding inverse demand function is

$$D^{-1}_{14}(d^{14}) = 10(15 - d^{14})\,.$$

The details of the computations can be seen in the accompanying Excel spreadsheet. In this document, I list the steps sequentially, intentionally not giving numbers to each step — I do this to emphasize the overall flow of the algorithm, and to not become overly attached to a specific numbering of the steps (since there are equivalent ways to implement the same algorithm). The Excel spreadsheet does assign iteration numbers to these steps.

- Initialize with the solution $d^{14} = 10$, $x_{13} = x_{34} = 10$ and all other link flows zero.
- Recalculating the travel times with these flows, the shortest path is $[1, 2, 4]$, with a cost $\kappa = 60$. So $TSTT = \mathbf{x} \cdot \mathbf{t} = 10 \times 60 + 10 \times 100 = 1600$, while $SPTT = \mathbf{d} \cdot \boldsymbol{\kappa} = 10 \times 60 = 600$. Thus $AEC = (1600 - 600)/10 = 100$. The current value of the demand function is $D^{14}(60) = 9$, so the total misplaced flow is 1.
- The new target demand is $d^*_{14} = D^{14}(60) = 9$, and the new target link flows are $x^*_{12} = x^*_{24} = 9$ and zero elsewhere.

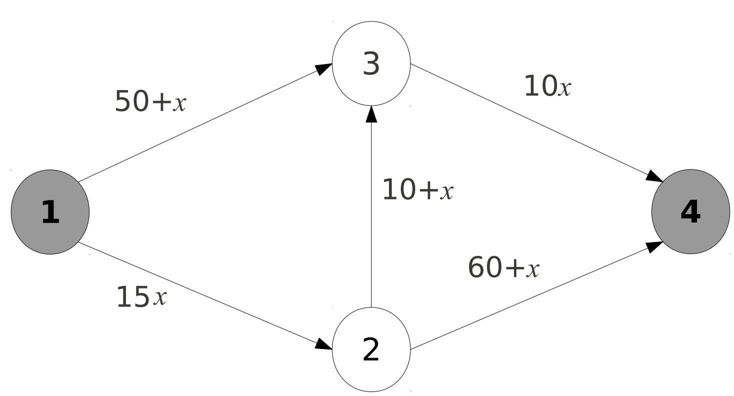

Figure 1: Network for worked-out example

- We choose λ to solve

$$t_{12}(\lambda x_{12}^* + (1-\lambda)x_{12})(x_{12}^* - x_{12}) + \cdots + t_{34}(\lambda x_{34}^* + (1-\lambda)x_{34})(x_{34}^* - x_{34}) + D_{14}^{-1}(\lambda d_{14}^* + (1-\lambda)d_{14})(d_{14}^* - d_{14}) = 0\,, \quad (1)$$

  namely, $\lambda = 0.420$.

- Taking a step of size λ gives a new demand of $d^{14} = 9.58$, and link flows $x_{12} = x_{24} = 3.78$, $x_{13} = x_{34} = 5.80$, and $x_{23} = 0$.

- Recalculating the travel times with these flows, the shortest path is $[1,3,4]$, with a cost $\kappa = 114$. So the new values of $TSTT$ and $SPTT$ (calculated with the latest flows) are 1115 and 1090, respectively. Thus $AEC = 2.61$. The current value of the demand function is $D^{14}(114) = 3.62$, so the total misplaced flow is 5.96.

- The new target demand is $d_{14}^* = D^{14}(114) = 3.62$, and the new target link flows are $x_{13}^* = x_{34}^* = 3.62$ and zero elsewhere.

- Solving equation (1) with the new $\mathbf{x}$, $\mathbf{d}$, $\mathbf{x}^*$, and $\mathbf{d}^*$ values gives $\lambda = 0.598$.

- Taking a step of size λ gives a new demand of $d^{14} = 6.01$, and link flows $x_{12} = x_{24} = 1.52$, $x_{13} = x_{34} = 4.50$, and $x_{23} = 0$.

- Recalculating the travel times with these flows, the shortest path is $[1,2,3,4]$, with a cost $\kappa = 77.7$. So the new values of $TSTT$ and $SPTT$ (calculated with the latest flows) are 575 and 468, respectively. Thus $AEC = 17.9$. The current value of the demand function is $D^{14}(77.7) = 7.23$, so the total misplaced flow is 1.21.

- The new target demand is $d_{14}^* = D^{14}(114) = 3.62$, and the new target link flows are $x_{12}^* = x_{23}^* = x_{34}^* = 3.62$ and zero elsewhere.

- Solving equation (1) with the new $\mathbf{x}$, $\mathbf{d}$, $\mathbf{x}^*$, and $\mathbf{d}^*$ values gives $\lambda = 0.187$.

- Taking a step of size λ gives a new demand of $d^{14} = 6.24$, and link flows $[x_{12}, x_{13}, x_{23}, x_{24}, x_{34}] = [2.59, 3.65, 1.35, 1.23, 5.01]$.

- Recalculating the travel times with these flows, the shortest path is $[1,2,4]$, with a cost $\kappa = 100.1$. So the new values of $TSTT$ and $SPTT$ (calculated with the latest flows) are 638 and 625, respectively. Thus $AEC = 2.19$. The current value of the demand function is $D^{14}(77.7) = 4.99$, so the total misplaced flow is 1.25.

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Continuing *ad libitum*, both the average excess cost and total misplaced flow will converge to zero (although, as seen in these steps so far, not necessarily monotonically).