Lighthill-Whitham-Richards traffic flow model

CE 392D

OUTLINE

1. Hydrodynamic model (LWR)
2. Shockwaves
3. Characteristics
4. Newell-Daganzo method

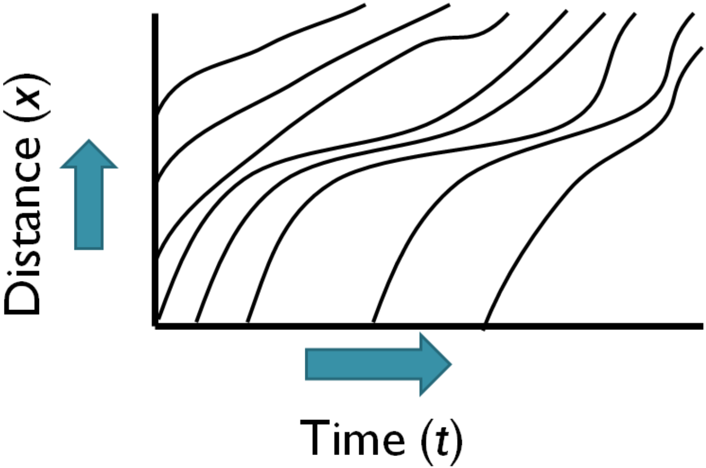
The Lighthill-Whitham-Richards model is commonly used for traffic flow.

There are three fundamental variables:

- *Flow* q, the rate at which vehicles pass a point.
- *Density* k, the spatial concentration of vehicles.
- *Speed* u, the average rate of travel.

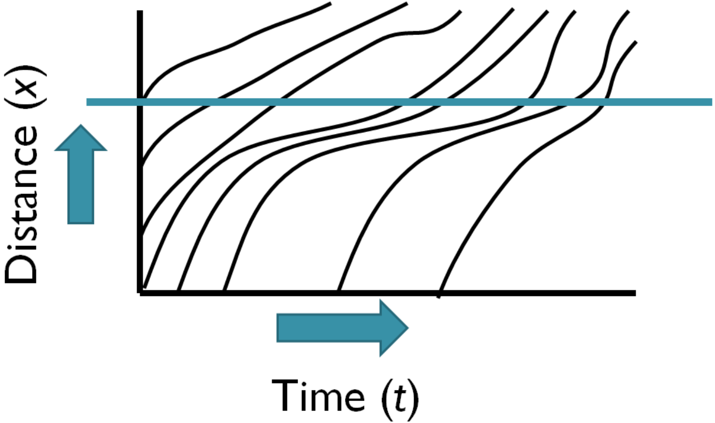
It is also useful to track N, the vehicle number or cumulative count at a point.

These quantities can be visualized on a *trajectory diagram*.

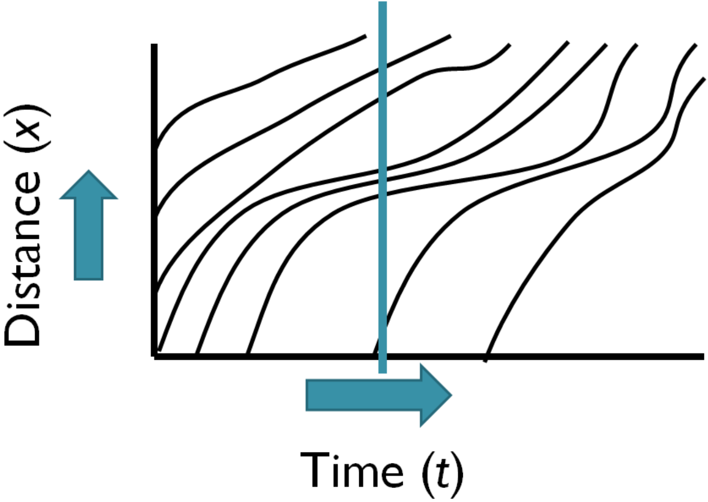


The trajectories represent contours of N.

The flow q is the rate trajectories cross a horizontal line.



The density k is the rate trajectories cross a vertical line.



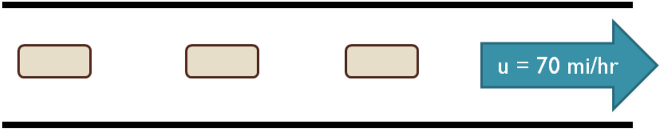
How is speed represented in a trajectory diagram?

In a link model, we are often given *initial conditions* (density values at $t = 0$) and *boundary conditions* (flow rates at $x = 0$, restrictions on flow rates at the downstream end.)

The task is to determine the outflows at the downstream end of the link; this may include finding N, q, k, and u at all points x and times t. We can treat these variables as functions of x and t.

How are these quantities related to each other?

I sit at the side of the road for one hour, while cars drive by at 70 mi/hr. If the density is 10 veh/mi, how many vehicles pass by?



k = 10 veh/mi

This is a demonstration of the *fundamental relationship* between speed, flow, and density:

$$q = uk$$

Think units: [veh/hr] = [mi/hr][veh/mi]

Flow and density are related to the cumulative counts:

$$q(x,t) = \frac{\partial N(x,t)}{\partial t}$$

$$k(x,t) = -\frac{\partial N(x,t)}{\partial x}$$

If N is twice continuously differentiable,

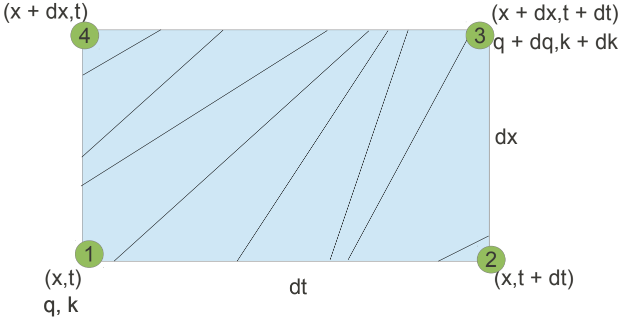
$$\frac{\partial^2 N}{\partial x \partial t} = \frac{\partial^2 N}{\partial t \partial x}$$

Therefore

$$\frac{\partial q}{\partial x} + \frac{\partial k}{\partial t} = 0$$

This is one way to derive the *conservation equation*. This equation holds everywhere these derivatives exist.

Another derivation of the conservation law



1. $N(x,t) = N_0$
2. $N(x,t+dt) = N_0 + q\ dt$
3. $N(x+dx,t+dt) = N_0 + q\ dt - (k+dk)\ dx$
4. $N(x+dx,t) = N_0 + q\ dt - (k+dk)\ dx - (q+dq)\ dt$
5. $N(x,t) = N_0 + q\ dt - (k+dk)\ dx - (q+dq)\ dt + k\ dx$

Therefore $\partial q/\partial x + \partial k/\partial t = 0$

How are flow and density related?

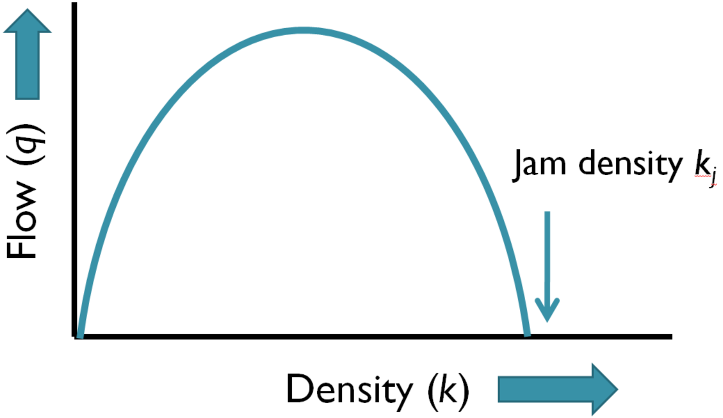
If $q = 0$, *either* $k = 0$ or $u = 0$.

If $u = 0$, then the density is at its maximum value k_j, the *jam density*.

So, $q = 0$ if $k = 0$ or $k = k_j$.

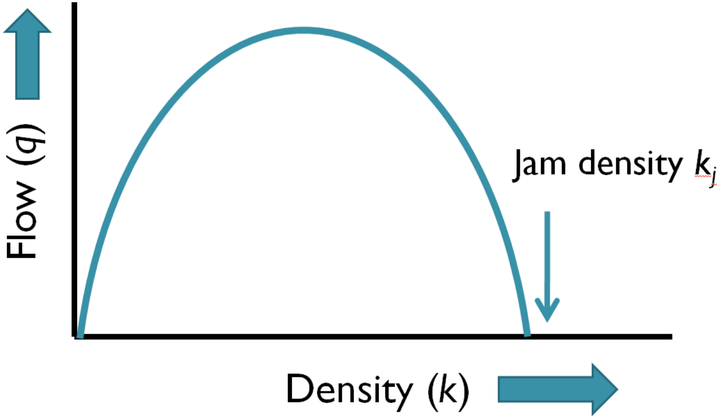
The major assumption in the LWR model is that q is a function of k alone. (k determines u, so $q(k) = u(k) \cdot k$)

Therefore, $q = Q(k)$ must be a concave function with zeros at $k = 0$ and $k = k_j$



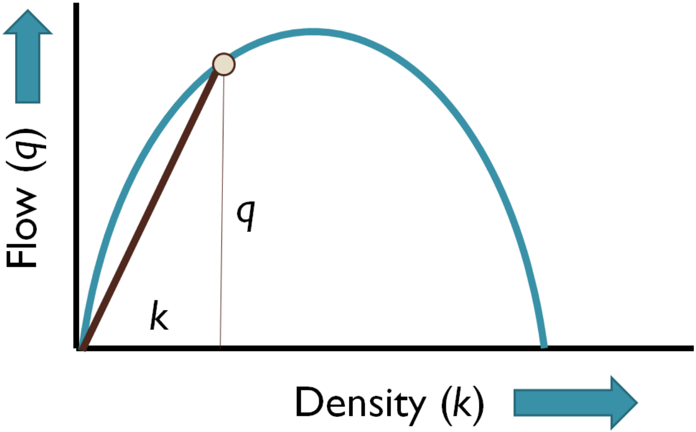
This is the *fundamental diagram*. The fundamental diagram can be calibrated to data, resulting in different traffic flow models.

Therefore, $q = Q(k)$ must be a concave function with zeros at $k = 0$ and $k = k_j$

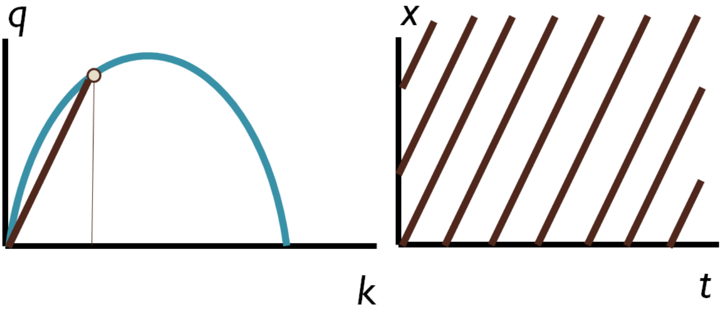


Also notice that there is some point at which flow is maximal. This maximum flow is the capacity q_{max}, which occurs at the critical density k_c.

We can get speed information from the fundamental diagram: $q = uk$, so $u = q/k$.



This is the same slope as on a trajectory diagram.



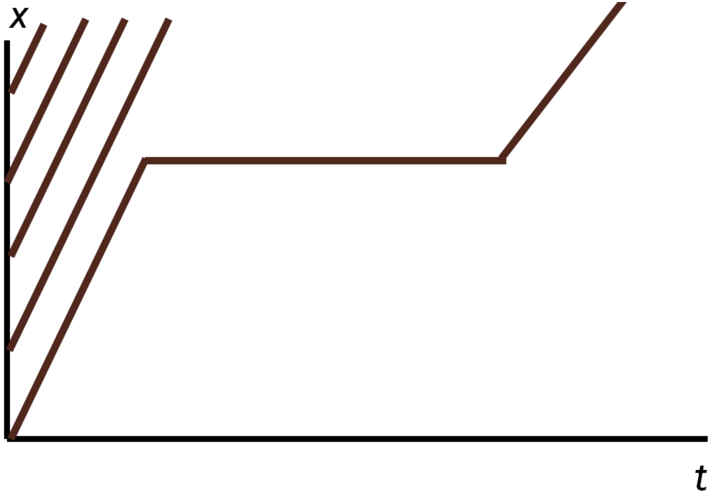
Some common fundamental diagrams...

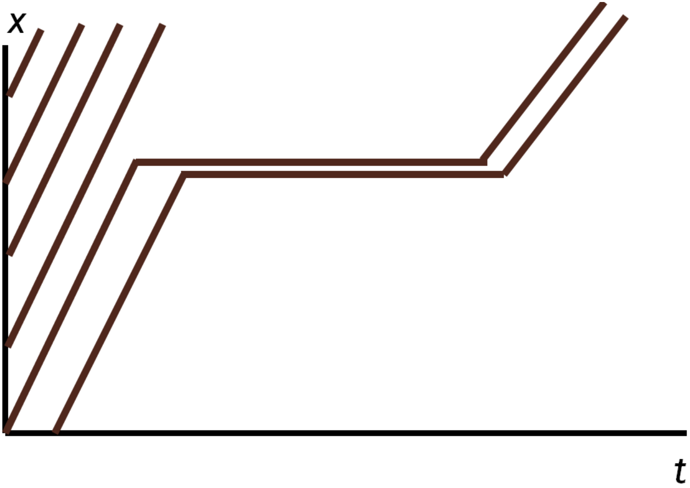
- Greenshields model: $q \propto k(k_j - k)$
- Greenberg model: $q \propto k \log(k_j/k)$
- Highway Capacity Manual
- Triangular: $q = \min\{u_f k, w(k_j - k)\}$
- Trapezoidal: $q = \min\{u_f k, q_{max}, w(k_j - k)\}$

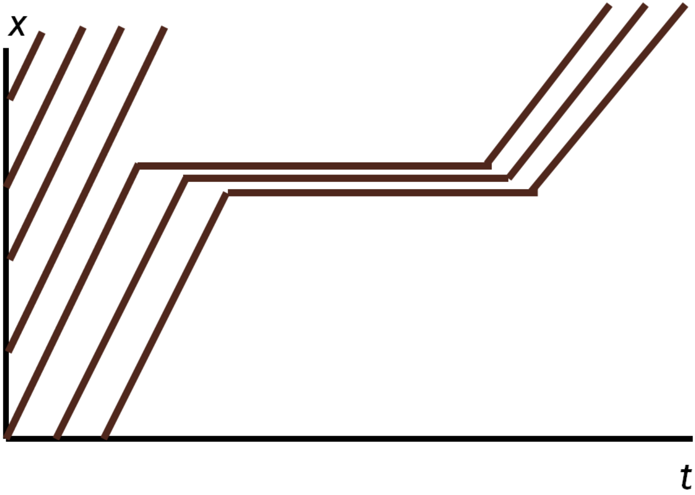
What happens if something interrupts this flow?

SHOCKWAVES

Let's say one car stops for a while. What happens to the next vehicles?




x
t


x
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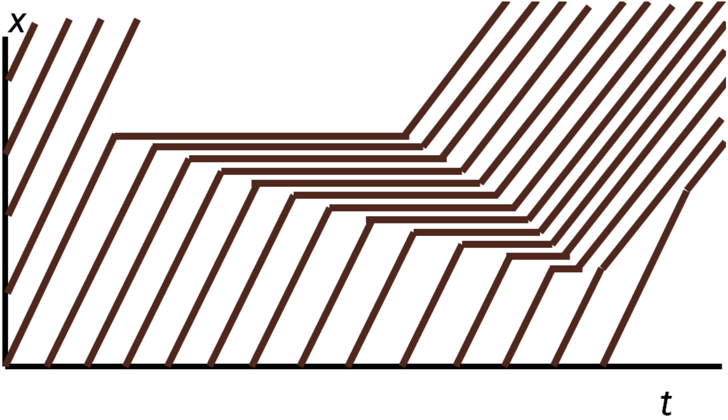
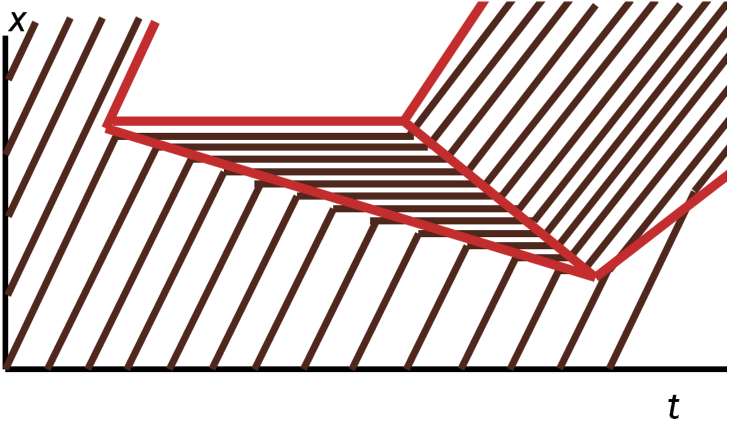
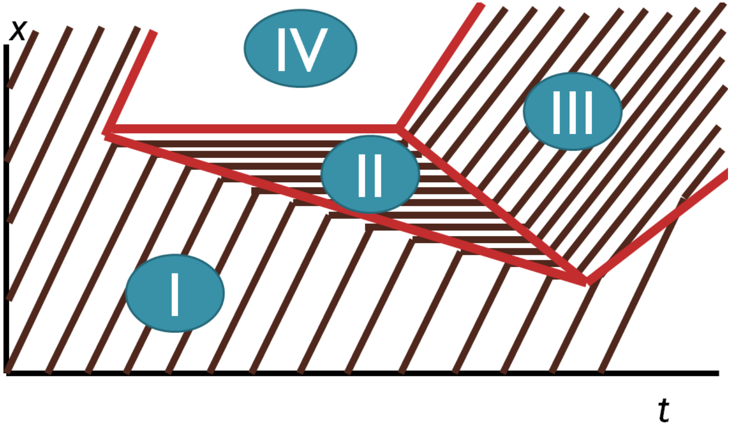

x
t

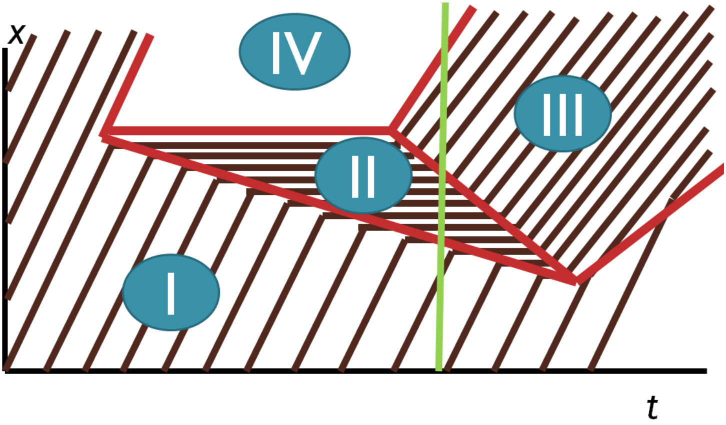
We can identify regions where density is continuous. The boundaries between these regions are *shockwaves.*

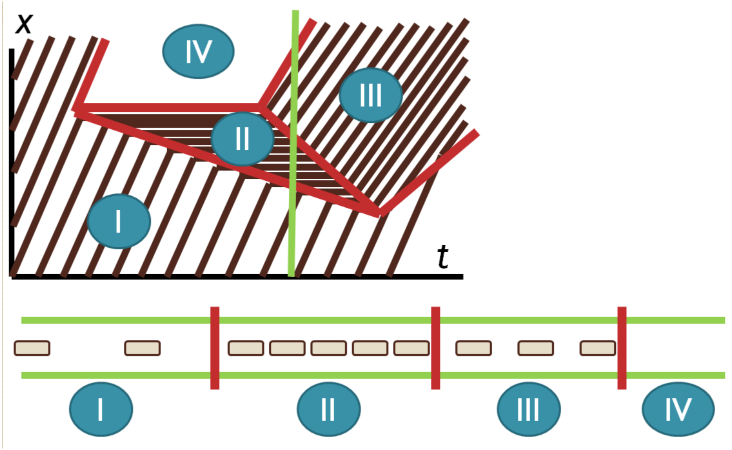


Let's label these regions I, II, III, and IV.



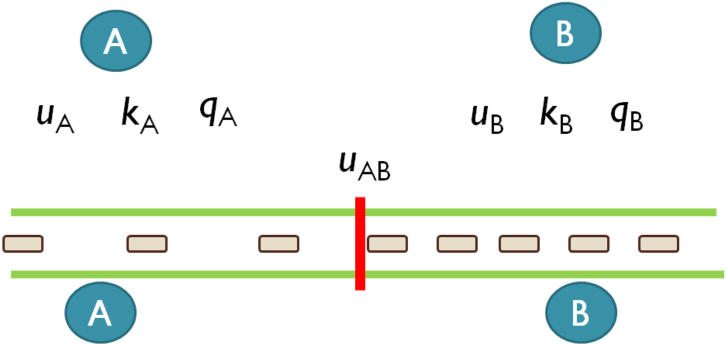
Let's look at a vertical slice of the trajectory diagram.



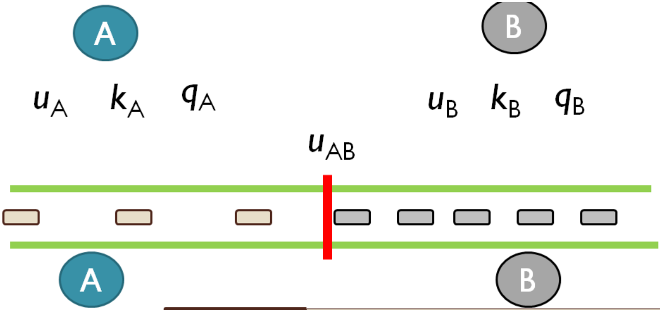


How quickly are these shockwaves moving? (In other words, when will the link fill up?)

Consider an arbitrary shockwave.



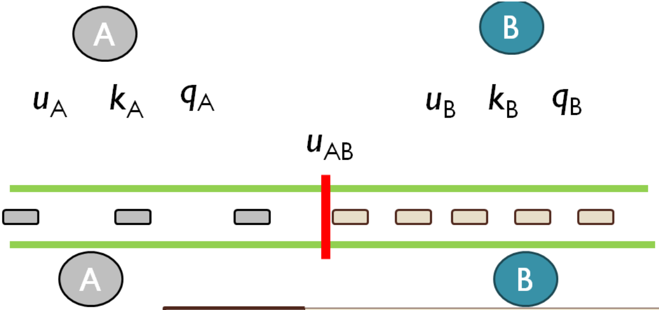
What rate are vehicles *entering* the shockwave?



Vehicle speed from the left relative to shockwave

$$q_{AB}^{\rightarrow|} = u_{AB}^{\rightarrow|} k_A = (u_A - u_{AB}) k_A$$

What rate are vehicles *leaving* the shockwave?



Vehicle speed from the right relative to shockwave

$$
\begin{aligned}
q_{AB}^{|\rightarrow} &= u_{AB}^{|\rightarrow} k_A \\
&= (u_B - u_{AB}) k_B
\end{aligned}
$$

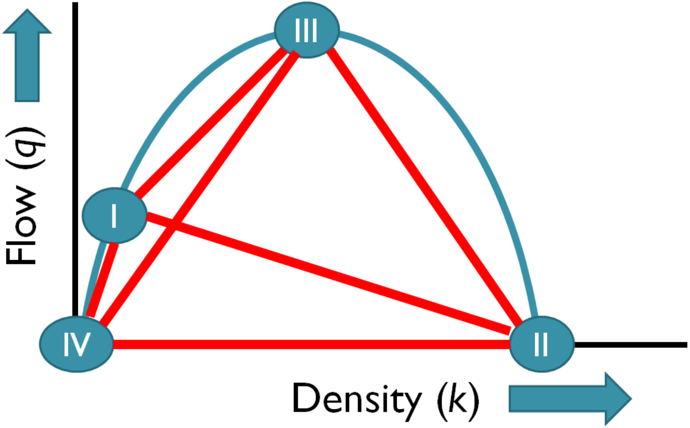
These flow rates should be identical, since vehicles are not appearing or disappearing at the shockwave. Thus

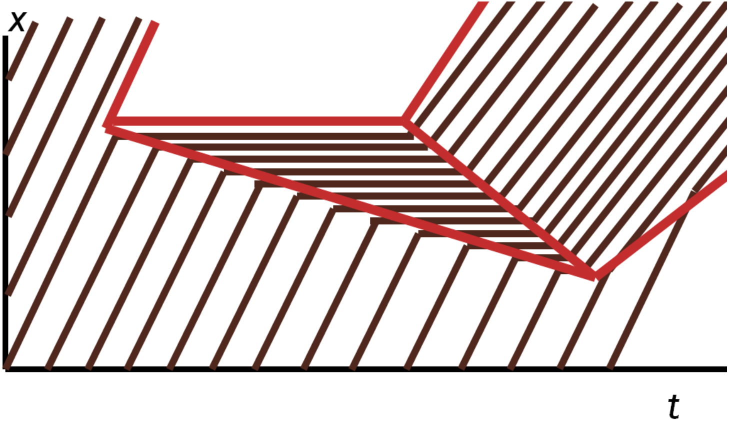
$$(u_A - u_{AB})k_A = (u_B - u_{AB})k_B$$

or

$$u_{AB} = \frac{q_A - q_B}{k_A - k_B}$$

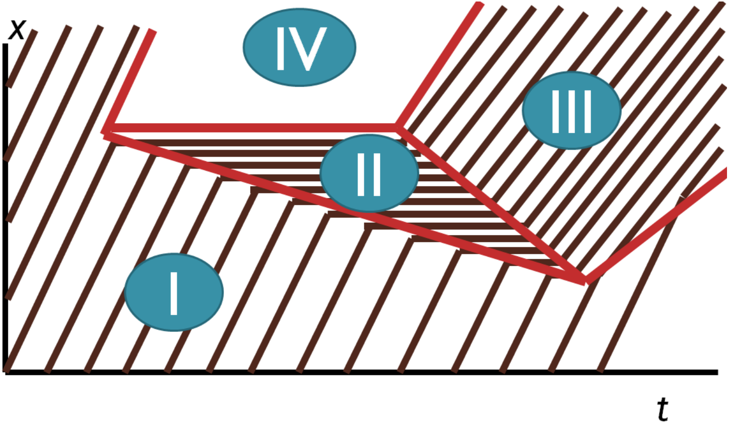
The slope of the shockwaves on a trajectory diagram is the slope of the line connecting the points on the fundamental diagram.




x
t

Example

Assume that $Q(k) = k(240 - k)/4$. (This implies $u_f = 60$, $k_j = 240$, $q_{max} = 3600$.)



In region 1, we have $k = 20$ (so $q = 1100$ and $u = 55$). Assume flow in region 3 is at capacity.

PARTIAL DIFFERENTIAL EQUATION FORMULATION

Solving the LWR model using shockwaves is tedious, difficult, and not readily implemented on computers. The LWR model can also be formulated as a system of partial differential equations:

$$q(x,t) - k(x,t)u(x,t) = 0 \tag{1}$$

$$q(x,t) - Q(k(x,t)) = 0 \tag{2}$$

$$\frac{\partial q}{\partial x} + \frac{\partial k}{\partial t} = 0 \tag{3}$$

Furthermore, the cumulative count can be calculated using the equation

$$N(x_2, t_2) - N(x_1, t_1) = \int_C q\ dt - k\ dx$$

The conservation law guarantees that this line integral is independent of path. Linear or piecewise linear paths are usually chosen for ease of integration.

CHARACTERISTICS

We want to use the conservation law to help solve LWR problems.

In the language of differential equations, we want to find functions $k(x, t)$ and $q(x, t)$ such that

1. Conservation is satisfied: $\partial q/\partial x + \partial k/\partial t = 0$
2. The fundamental diagram is satisfied: $q(x, t) = Q(k(x, t))$
3. Any boundary conditions are satisfied.

Luckily, these PDEs can usually be solved without too much difficulty.

The fundamental diagram implies that q is a function of k.

Therefore, the conservation law $\dfrac{\partial k}{\partial t} + \dfrac{\partial q}{\partial x} = 0$ can be rewritten

$$\frac{\partial k}{\partial t} + \frac{dq}{dk}\frac{\partial k}{\partial x} = 0$$

which is a PDE in $k(x, t)$ alone.

We can solve this PDE by looking for "characteristics," curves along which $k(x, t)$ is constant.

$$\frac{\partial k}{\partial t} + \frac{dq}{dk}\frac{\partial k}{\partial x} = 0$$

In this model, the characteristics are straight lines. Consider a line with slope dq/dk, so $dx = \frac{dq}{dk}dt$

If we move in this direction,

$$dk = \frac{\partial k}{\partial t}dt + \frac{\partial k}{\partial x}dx$$

$$dk = -\frac{dq}{dk}\frac{\partial k}{\partial x}dt + \frac{\partial k}{\partial x}\frac{dq}{dk}dt = 0$$

so $k(x,t)$ is constant along a line with slope dq/dk. This is called the *wave speed*.

So, if I know the value of $k(x,t)$ at any point (e.g., boundary condition), I know the value of $k(x,t)$ at all points along the line through (x,t) with slope $dq(x,t)/dk$...unless there is "something else" which interferes (i.e., a shockwave).

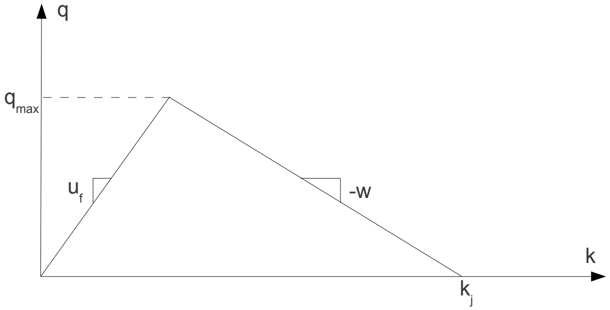
In the "uncongested" portion of the fundamental diagram, $dq/dk > 0$, so **uncongested states propagate downstream**.

In the "congested" portion of the fundamental diagram, $dq/dk < 0$, so **congested states propagate upstream**.

In other words, where there is no congestion, upstream conditions prevail. Where congestion exists, downstream conditions prevail.

NEWELL-DAGANZO METHOD

The Newell-Daganzo method is an easier alternative to solving the LWR model.



For now, assume a triangular fundamental diagram with only two wave speeds: u_f for the uncongested portion, and $-w$ for the congested portion.

Notice that in a triangular fundamental diagram, speed does not drop until density exceeds the critical density and congestion sets in.

The rough logic behind the method:

1. We want to calculate $k(x,t)$ or $N(x,t)$ at some point (x,t).
2. Either this point is congested or uncongested.
3. If congested, the wave speed is $-w$, so past conditions downstream will determine $k(x,t)$ and $N(x,t)$ here.
4. If uncongested, the wave speed is u_f, so past conditions upstream will determine $k(x,t)$ and $N(x,t)$ here.
5. Of these two possibilities, the correct solution is the one corresponding to the *lowest* $N(x,t)$ value.

If upstream conditions prevail, the $N(x,t)$ value based on the uncongested wave speed will be lower. If downstream conditions prevail, the $N(x,t)$ value based on the congested wave speed will be lower.

The major tools in the Newell-Daganzo method:

1. $$N(x_2, t_2) - N(x_1, t_1) = \int_C q\ dt - k\ dx$$
2. k (and therefore q) is constant along characteristics
3. Characteristics are straight lines, so $q\ dt - k\ dx$ is *constant*, so the integral is easy to evaluate.
4. With the simplified fundamental diagram, there are only two characteristic slopes possible

Change in vehicles along characteristic with positive slope

These characteristics reflect uncongested conditions, and have slope (wave speed) u_f.

$$\int_C q\ dt - k\ dx = \int_C \left(q - k \frac{dq}{dk} \right)\ dt$$

dq/dk is the wave speed, u_f.

However, u_f is also the traffic speed for the uncongested case, so $q = u_f k$ and

$$\int_C \left(q - k \frac{dq}{dk} \right)\ dt = 0$$

$N(x,t)$ is constant along forward-moving characteristics. In other words, if you move with the speed of uncongested traffic, you should observe no change in the cumulative vehicle count.

Change in vehicles along characteristic with negative slope

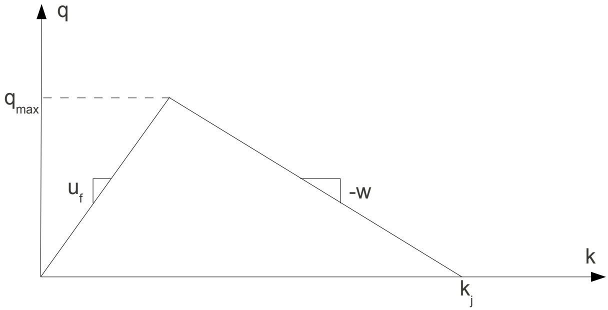
These characteristics reflect uncongested conditions, and have slope (wave speed) $-w$.

$$\int_C q\ dt - k\ dx = \int_C \left(\frac{q}{dq/dk} - k \right)\ dx$$

dq/dk is the wave speed, $-w$.

$$\int_C q\ dt - k\ dx = -\int_C (k + q/w)\ dx$$

From the fundamental diagram, $k + q/w = k_j$



So

$$\int_C q\ dt - k\ dx = -\int_C k_j dx = k_j(x_2 - x_1)$$

If you are moving at the backward wave speed, the cumulative vehicle count increases at the rate of the jam density.

Example

A link is 1 mile long, and has free flow speed 30 mph, backward wave speed 15 mph, and jam density 200 veh/mi. Vehicles enter upstream at a rate of 1200 veh/hr. Three minutes from now, the downstream traffic signal turns red. Four minutes from now, what is the cumulative count value at the midpoint of the link? Has the queue reached this point?

With a more general fundamental diagram, there are a larger number of possible characteristic speeds.

The problem then is to trace back all possible characteristics to a known point (initial condition, boundary condition, previously-solved value), determine the change in vehicle number, and select the smallest.

This can be framed as a *calculus of variations* problem.