LWR-based Link Models

CE 392D

OUTLINE

LWR AND LINK MODELS

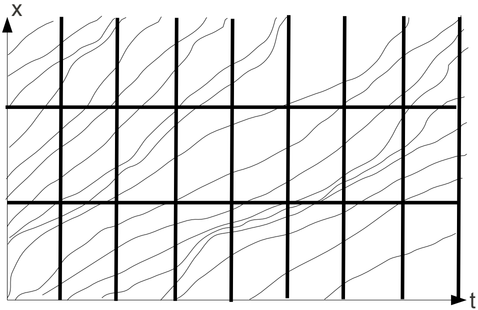
We will use the LWR model to develop link models which can represent congested behavior more accurately than point queue or spatial queue models.

The **cell transmission model** approximately solves the LWR partial differential equation system by discretizing in space and time.

The **link transmission model** exactly solves the LWR system using the Newell-Daganzo method.

CELL TRANSMISSION MODEL

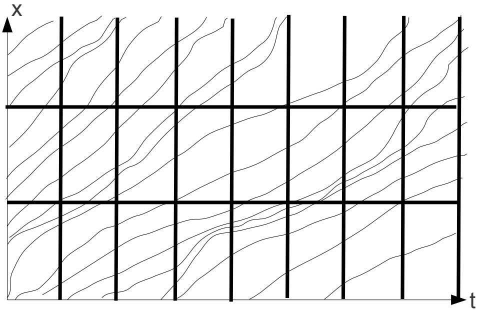
The cell transmission model is a discrete approximation to the LWR model



3	3	3	2	2	1	2	3
3	4	4	4	2	2	4	5
4	4	4	4	5	4	3	1

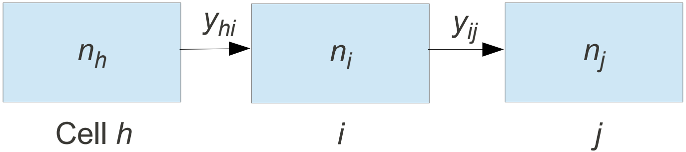
A roadway link is divided into “cells,” and we track the number of vehicles in each cell at discrete points in time.

The key idea in CTM is that *the length of a cell is the distance a vehicle would travel at free flow, in one time step*. Therefore, $\Delta x = u_f \Delta t$



3	3	3	2	2	1	2	3
3	4	4	4	2	2	4	5
4	4	4	4	5	4	3	1

In CTM, we keep track of the number of vehicles in each cell i at each time t : $n_i(t)$



If $y_{ij}(t)$ is the number of vehicles moving from cell i to cell j during time interval t, then

$$n_i(t+1) = n_i(t) + y_{hi}(t) - y_{ij}(t)$$

This is a discrete form of the conservation equation!

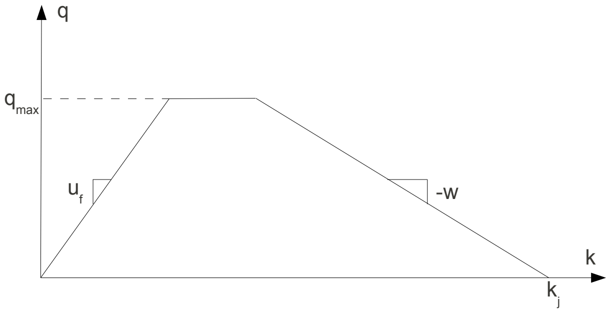
If we assume a uniform flow rate q between two cells during a time interval, and uniform density k in a cell during a time interval, then

$$y_{hi}(t) = q_{hi} \Delta t$$

and

$$n_i(t) = k \Delta x$$

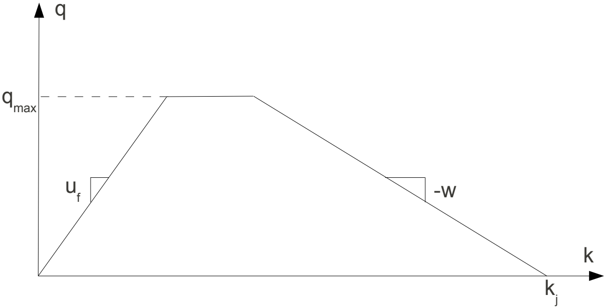
CTM was originally presented for a trapezoidal fundamental diagram.



This diagram is defined by four properties: the free-flow speed u_f, the capacity q_{max}, the jam density k_j, and the backward wave speed $-w$.

This fundamental diagram has the form

$$q = \min\{u_f k, q_{max}, w(k_j - k)\}$$



Therefore, the flow $y \approx q\Delta t$, so substituting the flow-density relation gives

$$y = \min\{u_f k\Delta t, q_{max}\Delta t, w(k_j - k)\Delta t\}$$

$$y = \min\left\{n_i(t), q_{max}\Delta t, \frac{w}{v}(N_i - n_i(t))\right\}$$

where $N_i = k_j\Delta x$ is the maximum number of vehicles which can fit into cell i

$$y = \min\left\{n_i(t), q_{max}\Delta t, \frac{w}{v}(N_i - n_i(t))\right\}$$

So, there are three possibilities:

1. $y = n_i(t)$. This is the *uncongested* case, so the wave propagates *downstream*. Every vehicle currently in the cell advances to the next.
2. $y = q_{max}\Delta t$. This is the *full-capacity* case. Vehicles leave the cell at capacity.
3. $y = w/v(N_i - n_i(t))$. This is the *congested* case, so the wave propagates *upstream*. The number of vehicles which can *enter* the cell is restricted by the number of vehicles which fit at jam density.

One advantage of CTM is that it can be described in a simple way, which does not require sophisticated knowledge of traffic flow theory to understand.

The CTM can also be cast in terms of the partial differential equation formulation of the LWR model:

- It is an example of an *explicit* solution method, where we approximate the differential equation by finite differences on a grid, and solve in forward order of time.
- The relationship between cell length and the time step corresponds to the *Courant-Friedrich-Lewy condition* for stability of explicit solution methods.

CTM provides information to a node model for calculating boundary inflows and outflows:

$$S = \min\left\{n(t), q_{max}\Delta t\right\}$$

calculated at the cell at the downstream link end; and

$$R = \min\left\{q_{max}\Delta t, \frac{w}{v}(N - n(t))\right\}$$

calculated at the cell at the upstream link end.

Recall the sending flow is the number of vehicles which are ready to leave the link in the next time step; the receiving flow is the number of vehicles which can be accommodated on the link in the next time step.

Example

Assume that a link has three cells; at most 10 vehicles can move between cells in one time step; at most 30 vehicles can fit into any cell at one time; and $\delta = 2/3$.

The desired input to the link is given as $d(t)$; assume that if these vehicles are unable to enter the link, they queue upstream and will enter as soon as possible.

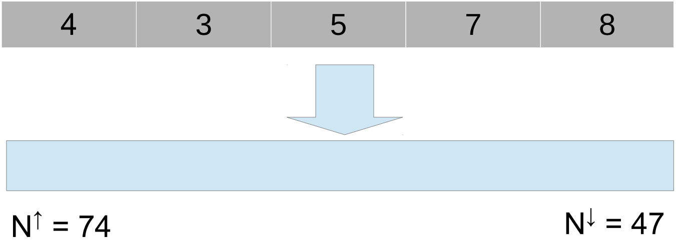
At the downstream end of the link, there is a red light which will turn green at $t = 10$ (and stay green).

				Cell 0		Cell 1		Cell 2		
t	$d(t)$	$R(t)$	$y(0,t)$	$N(0,t)$	$y(1,t)$	$N(1,t)$	$y(2,t)$	$N(2,t)$	$S(t)$	$y(3,t)$
0	10	10	10	0	0	0	0	0	0	0
1	10	10	10	10	10	0	0	0	0	0
2	10	10	10	10	10	10	10	0	0	0
3	10	10	10	10	10	10	10	10	10	0
4	10	10	10	10	10	10	6.7	20	10	0
5	9	10	9	10	10	13.3	2.2	26.7	10	0
6	8	10	8	9	5.9	21.1	0.7	28.9	10	0
7	7	10	7	11.1	2.5	26.3	0.2	29.6	10	0
8	6	9.6	6	15.6	1	28.5	0.1	29.9	10	0
9	5	6.3	5	20.6	0.4	29.4	0	30	10	0
10	4	3.2	3.2	25.2	0.1	29.8	0	30	10	10
11	3	1.2	1.2	28.3	0.1	29.9	6.7	20	10	10
12	2	0.4	0.4	29.4	4.5	23.3	8.9	16.7	10	10
13	1	3.1	3.1	25.3	7.4	18.9	9.6	15.6	10	10
14	0	6	1.3	21.0	8.9	16.7	9.9	15.2	10	10
15	0	10	0	13.4	9.5	15.7	10	15.1	10	10
16	0	10	0	3.9	3.9	15.3	10	15	10	10
17	0	10	0	0	0	5.8	9.2	15	10	10
18	0	10	0	0	0	0	0	14.2	10	10
19	0	10	0	0	0	0	0	4.2	4.2	4.2
20	0	10	0	0	0	0	0	0	0	0

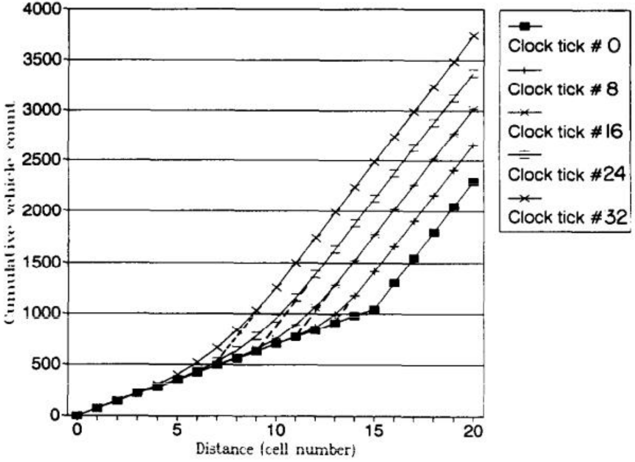
t	$N(0,t)$	$N(1,t)$	$N(2,t)$
0	0	0	0
1	10	0	0
2	10	10	0
3	10	10	10
4	10	10	20
5	10	13.3	26.7
6	9	21.1	28.9
7	11.1	26.3	29.6
8	15.6	28.5	29.9
9	20.6	29.4	30
10	25.2	29.8	30
11	28.3	29.9	20
12	29.4	23.3	16.7
13	25.3	18.9	15.6
14	21	16.7	15.2
15	13.4	15.7	15.1
16	3.9	15.3	15
17	0	9.2	15
18	0	0	14.2
19	0	0	4.2
20	0	0	0

LINK TRANSMISSION MODEL

The link transmission model is another solution scheme for the LWR model that further reduces the number of calculations needed, compared to CTM.

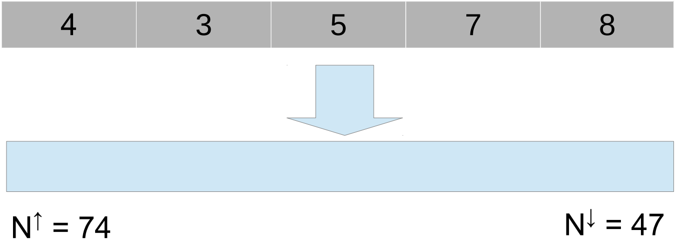


CTM maintains the count of vehicles at each cell within a network; LTM only tracks the cumulative counts $N^{\uparrow}$ and $N^{\downarrow}$ at the upstream and downstream ends.



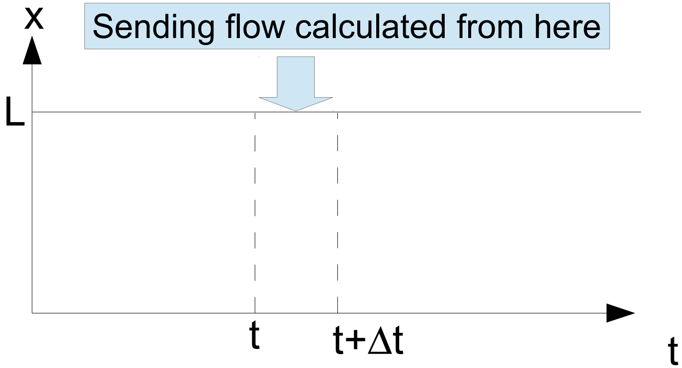
Moreoever, it can avoid the "shock spreading" effect witnessed in CTM when $w < u_f$.

LTM is based on the Newell-Daganzo method, applied *only* to the upstream and downstream ends of a link. Like CTM it is a discrete model which calculates these values at a timestep Δt.



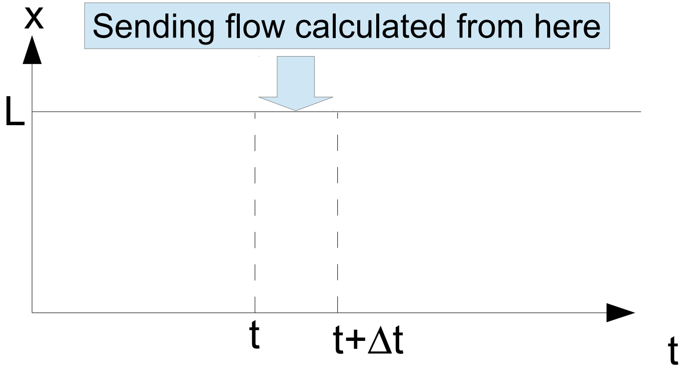
The goal is to calculate sending and receiving flows; by looking only at the ends of links, there is only *one* relevant wave speed.

At the *downstream* end of the link, we calculate the sending flow based on the uncongested wave speed, looking to the upstream end in the past to see how many vehicles can leave.



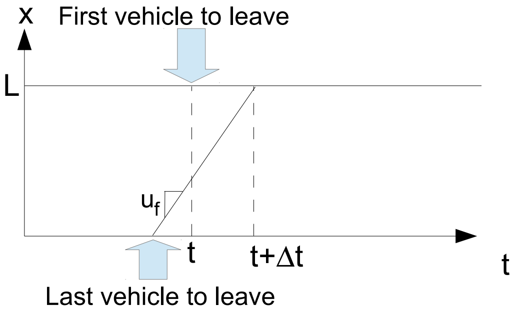
$$S(t) = \min\{N^{\uparrow}(t - L/u_f + \Delta t) - N^{\downarrow}(t), q_{max}\Delta t\}$$

We are using the same convention with discrete time indices:



$S(t)$ refers to the sending flow *during* the time interval $(t + \Delta t)$. $N^{\uparrow}(t)$ is the value of the upstream cumulative count at the *exact* time t.

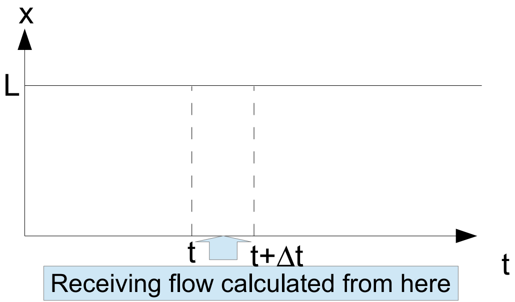
At the *downstream* end of the link, we calculate the sending flow based on the uncongested wave speed, looking to the upstream end in the past to see how many vehicles can leave.



$$S(t) = \min\{N^{\uparrow}(t - L/u_f + \Delta t) - N^{\downarrow}(t), q_{max}\Delta t\}$$

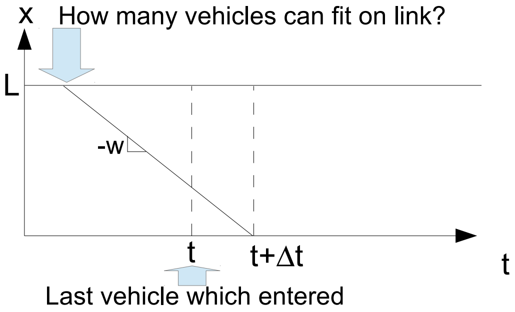
The intuition is the same as the with CTM: the sending flow is primarily based on the number of vehicles currently on the link.

At the *upstream* end of the link, we calculate the receiving flow based on the congested wave speed, looking to the downstream end in the past to see how many vehicles can enter (taking into account a queue from current or past conditions)



$$R(t) = \min\{N^{\downarrow}(t - L/w + \Delta t) + k_j L - N^{\uparrow}(t), q_{max}\Delta t\}$$

$$R(t) = \min\{N^{\downarrow}(t - L/w + \Delta t) + k_j L - N^{\uparrow}(t), q_{max}\Delta t\}$$



Given these sending and receiving flows, links in series, merges, and diverges function the same as in point queues or CTM.

$$S(t) = \min\{N^{\uparrow}(t - L/u_f + \Delta t) - N^{\downarrow}(t), q_{max}\Delta t\}$$

$$R(t) = \min\{N^{\downarrow}(t - L/w + \Delta t) + k_j L - N^{\uparrow}(t), q_{max}\Delta t\}$$

Example

Assume that a link has a free-flow time of 3 steps; a backward wave time of 4 steps; a capacity of 10; and that at most 90 vehicles can fit into the link at one time.

The desired input to the link is given as $d(t)$; assume that if these vehicles are unable to enter the link, they queue upstream and will enter as soon as possible.

At the downstream end of the link, there is a red light which will turn green at $t = 10$ (and stay green).

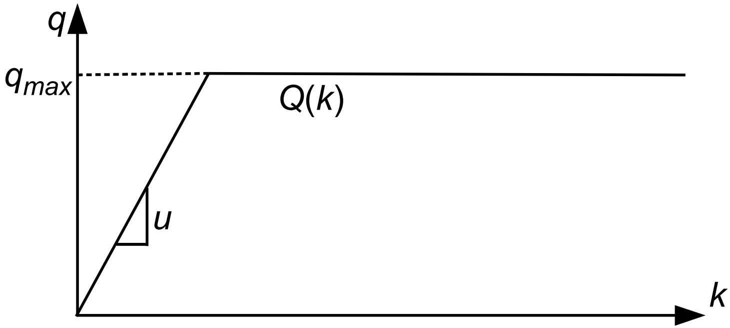
t	$d(t)$	$R(t)$	Inflow	$N^{\uparrow}(t)$	$N^{\downarrow}(t)$	$S(t)$	Outflow	Vehicles on link
0	10	10	10	0	0	0	0	0
1	10	10	10	10	0	0	0	10
2	10	10	10	20	0	0	0	20
3	10	10	10	30	0	10	0	30
4	10	10	10	40	0	10	0	40
5	9	10	9	50	0	10	0	50
6	8	10	8	59	0	10	0	59
7	7	10	7	67	0	10	0	67
8	6	10	6	74	0	10	0	74
9	5	10	5	80	0	10	0	80
10	4	5	4	85	0	10	10	85
11	3	1	1	89	10	10	10	79
12	2	0	0	90	20	10	10	70
13	1	0	0	90	30	10	10	60
14	0	10	5	90	40	10	10	50
15	0	10	0	95	50	10	10	45
16	0	10	0	95	60	10	10	35
17	0	10	0	95	70	10	10	25
18	0	10	0	95	80	10	10	15
19	0	10	0	95	90	5	5	5
20	0	10	0	95	95	0	0	0

POINT QUEUES, SPATIAL QUEUES, AND LWR

The point and spatial queue models can be seen as special cases of LWR-based link models, with particular fundamental diagrams.

(If you want more practice with the Newell-Daganzo method, try deriving the point queue sending/receiving flow expressions for these fundamental diagrams.)

The PQ model is equivalent to assuming the following fundamental diagram with CTM or LTM:



The SQ model is equivalent to assuming the following fundamental diagram with CTM or LTM: