19

FIRST-ORDER MACROSCOPIC TRAFFIC FLOW MODELS: INTERSECTION MODELING, NETWORK MODELING

Jean-Patrick Lebacque, INRETS-GRETIA2, France, and Megan M. Khoshyaran, ETC Economics Traffic Clinic, France

INTRODUCTION

The LWR (Lighthill and Whitham 1955, Richards 1956) model is currently the object of active research efforts, owing to the following facts: it is simple, easily calculable both analytically and numerically, and it recaptures with a reasonable precision many traffic phenomena, and models adequately many traffic situations. It has been implemented in several discretized models, FREFLOW Payne 1971, METANET Messner and Papageorgiou 1990, METACOR Elloumi et al., NETCELL Daganzo 1995, Lo 1999 for instance. Still many improvements are required, among which boundary and intersection modeling are prominent, because they constitute keys to: model identification and calibration using detector data, modeling of large and complex networks, applications to traffic management, such as ramp metering, speed control, dynamic reactive assignment, better understanding of capacity drop, hysteresis.

From a methodological point of view, the key to building macroscopic traffic flow models for networks is definition of proper boundary conditions. Since the LWR model results in a system of conservation laws, the analysis of the Riemann problem provides the main tool for defining boundary conditions.

The outline of the paper is the following. After a brief review of the literature, the equivalence between supply/demand boundary conditions and classical boundary conditions derived from the vanishing viscosity approach to conservation laws is proven. It is then shown that the simple mathematical intersection models of Holden and Risebro 1995 and Coclite and Piccoli 2002 can be considerably simplified within the supply-demand framework. Not all combinations of intersection supply and demand yield consistent intersection models, and a selection criterion is derived, the invariance principle. Two classes of point wise intersection models are introduced. One is based on an optimization principle of integrated supply and demand functions. The second is based on intersection equilibrium models in which the intersection is endowed with basic physical properties: storage capacity, maximum through flow, etc... Both approaches are shown to be equivalent in the case of merges and diverges, and to recapture previous models. A simple merge model is analyzed, and compared with field measurements. Finally, the boundary conditions of the FIFO multicommodity LWR model are analyzed, and the intersection models developed in the paper are combined with this model in order to yield a full network traffic flow model.

A BRIEF REVIEW OF THE LITERATURE

The LWR model is expressed by a single conservation law:

(1) $$\frac{\partial K}{\partial t}+\frac{\partial}{\partial x}Q_e(K,x)=0$$

with: x, t : the position and time, Q : the flow, K : the density, V : the speed, $Q_e\ (K,x)$: the equilibrium flow (fundamental diagram). $V_e\ (K,x)$ denotes the equilibrium speed-density relationship.

Link boundary conditions for such hyperbolic systems of conservation laws can be found in the mathematical literature, introduced by the pioneering works of Bardos-LeRoux-Nédélec (BLN) 1979 who used the viscosity method and Dubois-LeFloch (DL) (Dubois and Lefloch 1988) who relied on the Riemann problem approach. These two approaches are known to be equivalent in the scalar case, a result which applies to the LWR model. Both yield existence and unicity of entropy solutions under mild hypotheses in the scalar 1-D case. Further work by Otto 1992 perfected the BLN approach. The reader is referred to Kröner 1997.

Mathematicians have been showing some interest in the problem of point wise intersection modeling for the LWR model; see papers by Holden and Risebro 1995, Coclite-Piccoli 2002, Klar and Herty 2004 for instance. The resulting intersection models still lack realism.
In the field of transportation relatively little research effort has been devoted to the topic of boundary conditions, and mainly for the LWR model: Lebacque 1996, Lebacque and Khoshyaran 2002, Nelson and Kumar 2004.

In order to yield intersection models, link boundary conditions must be combined Buisson et al. 1995-1996, Lebacque and Khoshyaran 2002. Some pointwise intersection models have been proposed in the past (Lebacque 1984, Lebacque 1996, Daganzo 1995, Lebacque and Khoshyaran 2002, Jin and Zhang 2002). The difficulty of specifying boundary conditions has limited the scope of these earlier efforts, mainly limited to discretized models.

BOUNDARY CONDITIONS AND INTERSECTION MODELING

The supply-demand boundary conditions

From the fundamental diagram two equilibrium function can be constructed, the *equilibrium supply* Σ_e and *demand* Δ_e functions. The construction is illustrated by Figure 1 below. At a given point, the *local supply and demand* are defined as:

(2) $\Sigma(x,t)=\Sigma_e(K(x+,t),x+)$ and $\Delta(x,t)=\Delta_e(K(x-,t),x-)$

These quantities can be interpreted as the *greatest possible inflow* resp. *outflow* at any given location x. The symbols +, - in equation (2) represent right-hand side (resp. left-hand side) limits. Flow must be less than both supply and demand. The entropy solution of (1) is locally flow maximizing, Lebacque 1996. Thus the entropy solution can be characterized by:

(3) $Q(x,t)=\mathrm{Min}[\Sigma(x,t),\Delta(x,t)]$

a formula which will be referred to as the **min formula** in the rest of the paper.

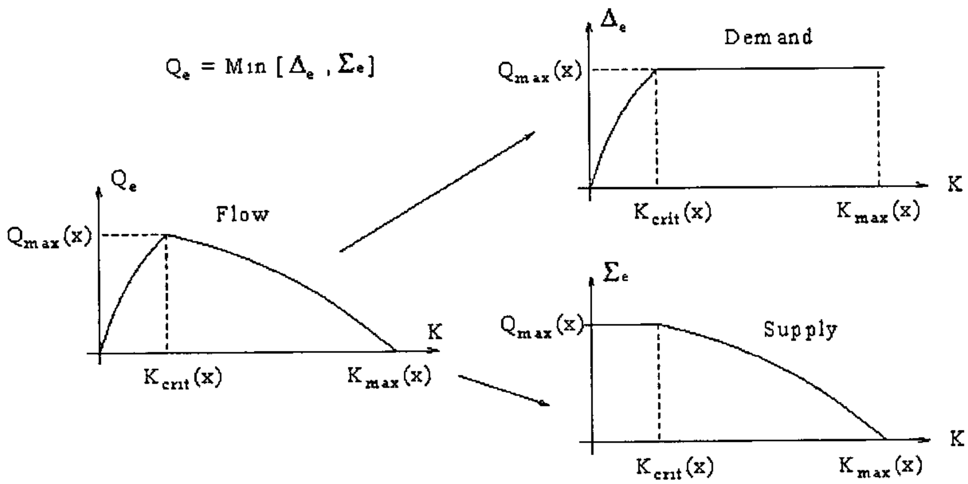


Figure 1: Fundamental diagram, equilibrium supply and demand functions

Let us now consider traffic flow on a link. Upstream boundary data is upstream demand $\Delta_u(t)$, the downstream boundary data is the downstream supply $\Sigma_d(t)$. Given the link supply $\Sigma(a,t)=\Sigma_e(K(a+,t),a)$ and demand $\Delta(b,t)=\Delta_e(K(b-,t),b)$, we apply the *min formula* (3) to derive the link inflow $Q(a,t)$ and link outflow $Q(b,t)$: The upstream demand and downstream supply determine the traffic inside the link.

(4) $Q(a,t)=\mathrm{Min}[\Delta_u(t),\Sigma(a,t)]$ and $Q(b,t)=\mathrm{Min}[\Delta(b,t),\Sigma_d(t)]$

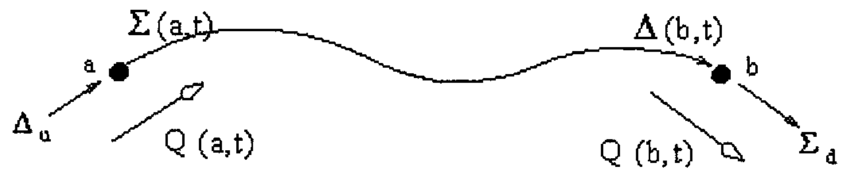


Figure 2: Supply and Demand data for link [a,b]

The density at the boundaries is given by:

$$(5)\ K(a,t+)=\begin{bmatrix}\Delta_e^{-1}(\Delta_u(t)) & \text{if} & \Delta_u(t)<\Sigma(a,t)\\ K(a,t) & \text{if} & \Delta_u(t)\geq\Sigma(a,t)\end{bmatrix}\quad K(b,t+)=\begin{bmatrix}\Sigma_e^{-1}(\Sigma_d(t)) & \text{if} & \Sigma_d(t)<\Delta(b,t)\\ K(b,t) & \text{if} & \Sigma_d(t)\geq\Delta(b,t)\end{bmatrix}$$

BLN (Bardos-LeRoux-Nédélec) boundary conditions. Equivalence with Supply/Demand boundary conditions.

The BLN boundary data is the prescription on the boundary of a *density-like* quantity A. More specifically, Bardos LeRoux Nédélec prove that equation (1) (and more generally scalar conservation equations) admits a unique solution on a link $D\overset{def}{=}[a,b]$ given an initial density on the link D and such that the density at the boundary $\partial D\overset{def}{=}\{a,b\}$ relates at all positive times to the boundary data A according to:

$$(6)\qquad \begin{gathered}\{\operatorname{sgn}[K(x,t)-\kappa]-\operatorname{sgn}[A(c,t)-\kappa]\}\,[Q_e(K(c,t),c)-Q_e(\kappa)].n(c)\geq 0\\ \forall c\in\partial D=\{a,b\}\ \text{ and }\ \forall\kappa\geq 0\end{gathered}$$

The symbol sgn represents the sign function, $n(c)$ is the normal to the domain boundary at c, that is $n(a)$=-1 and $n(b)$=1 since (1) is 1-D (Figure 3).

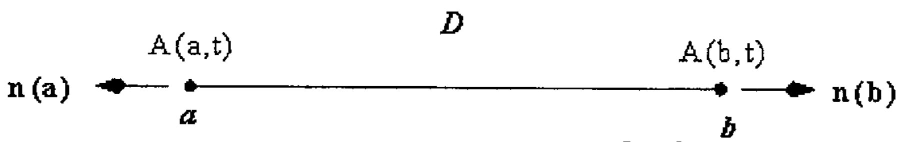


Figure 3: BLN data for a link $D=[a,b]$

This boundary condition (6) is obtained by analyzing the viscosity solutions of (1), and generalizing standard boundary conditions of the Dirichlet type for parabolic equations. The reader is referred to Kröner 1997 chapter 6, as well as to Otto 1992.

The BLN boundary condition (6) at the link entry point can be shown (see Lebacque 2003) to be equivalent to

$$(7)\qquad K\in\{A\}\cup[A^*,K_{max}]\ \text{if}\ A\leq K_{crit}\ ,\quad K\in[K_{crit},K_{max}]\ \text{if}\ A\geq K_{crit}$$

that is to say that the *upstream BLN data A is actually equivalent to the demand data* $\Delta_e(A)$. The conjugate A^* of A is such that $Q_e(A^*)=Q_e(A)$ and $A\neq A^*$. Proof: see Appendix.

Downstream BLN boundary conditions can be shown (see Lebacque 2003) to be equivalent to

$$(8)\qquad K\in[0,K_{crit}]\ \text{if}\ A\leq K_{crit}\ ,\quad K\in\{A\}\cup[0,A^*]\ \text{if}\ A\geq K_{crit}$$

that is to say that the *downstream BLN data A is equivalent to the demand data* $\Sigma_e(A)$. This result is of course symmetrical to the result for upstream boundary conditions.

Pointwise intersections.

Both Holden and Risebro 1995 and Coclite and Piccoli 2002 aim at solving a generalized Riemann problem at the intersection. Initial data are the densities K_{i0} , K_{j0} , which are assumed uniform on semi-infinite upstream links $[i]$ and downstream links $[j]$. The main problem addressed by Holden and Risebro is: which are the densities K_i , K_j and flows $Q_i = Q_e(K_i)$, $R_j = Q_e(K_j)$, at the node.

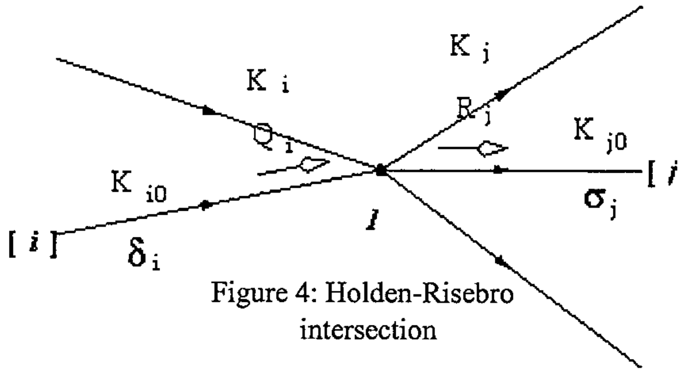


Figure 4: Holden-Risebro intersection

In both approaches, the following constraints are applied to the K_i , K_j :

$$
(9)\quad \left[\begin{array}{l}
\begin{cases} K_i \in \left[K_{i,crit}, K_{max}\right] & \text{if } K_{i0} \geq K_{i,crit} \\ K_i \in \{K_{i0}\} \cup \left[K_{i0}^*, K_{i,crit}\right] & \text{if } K_{i0} \leq K_{i,crit} \end{cases} \\
\begin{cases} K_j \in \left[0, K_{j,crit}\right] & \text{if } K_{j0} \leq K_{j,crit} \\ K_i \in \{K_{j0}\} \cup \left[0, K_{i0}^*\right] & \text{if } K_{i0} \geq K_{i,crit} \end{cases}
\end{array}\right.
$$

(9) expresses the BLN boundary conditions between K_{i0} (boundary data in the BLN sense) and K_i , and between K_{j0} (boundary data in the BLN sense) and K_j respectively. The reader will observe the similitude with (7) and (8). In the supply-demand framework, upstream boundary data is demand $\Delta_e(K_{i0})$ and downstream boundary data is supply $\Sigma_e(K_{j0})$. Thus (9) is equivalent to the simpler condition:

$$
(10)\quad \left[\begin{array}{ll}
Q_i \leq \Delta_e(K_{i0}) \overset{def}{=} \delta_i \quad \text{and} & \begin{cases} K_i = \Sigma_e^{-1}(Q_i) & \text{if } Q_i < \delta_i \\ K_i = K_{i0} & \text{if } Q_i = \delta_i \end{cases} \\
R_j \leq \Sigma_e(K_{j0}) \overset{def}{=} \sigma_j \quad \text{and} & \begin{cases} K_j = \Delta_e^{-1}(Q_j) & \text{if } R_j < \sigma_j \\ K_j = K_{j0} & \text{if } R_j = \sigma_j \end{cases}
\end{array}\right.
$$

Equation (10) expresses

- that the intersection through flows should be less than upstream demands and downstream supplies respectively
- that the densities are equal to the initial densities if the flow is not constrained by upstream demands and downstream supplies, and are defined by these constraints otherwise (see Figure 5).

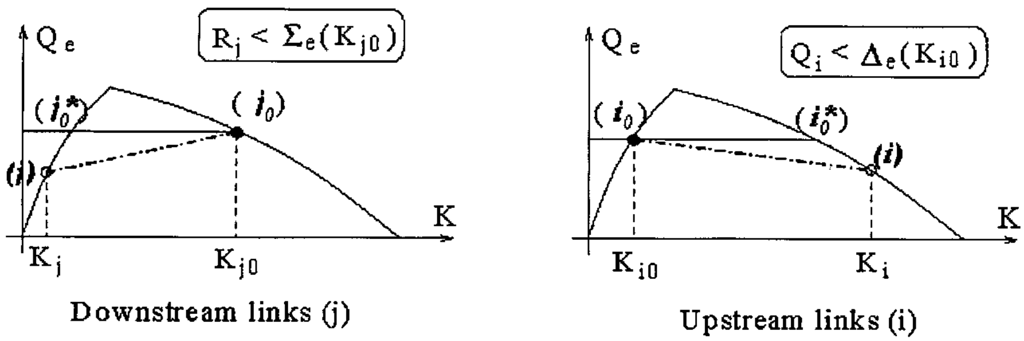


Figure 5: Admissible node states, for downstream and upstream links

If traffic states $(i)\stackrel{def}{=}(K_i,Q_i)$ and $(j)\stackrel{def}{=}(K_j,R_j)$ satisfy the above conditions (9) or (10), it is clear that on each link [i] or [j], the K_{i0} - K_i resp. $K_j - K_{j0}$ discontinuities propagate in the right direction, with <0 resp. >0 speed. This essential fact is demonstrated by the figure 5.

Thus the basic variables of the generalized Riemann problem for a point wise intersection are the node inflows Q_i and outflows R_j. The constraints that apply to these variables are posivity constraints, conservation constraints, and constraints resulting from (10):

$$\text{(11)} \qquad \left[\begin{array}{l} 0 \le Q_i \le \delta_i \quad \forall i, \quad \text{and} \quad 0 \le R_j \le \sigma_j \quad \forall j \\ \sum_i Q_i = \sum_j R_j \end{array}\right.$$

The Holden-Risebro and Coclite-Piccoli approaches resolve the under determinacy of (11) by optimizing a criterion subject to the constraints (11). Holden and Risebro propose to maximize a concave function of the Q_i and R_j . A similar idea was developed in Lebacque and Khoshyaran 2002, but based on the zone concept defined for nodes in the STRADA project (Buisson et al. 1995-1996). Coclite and Piccoli propose to maximize the total through flow of the node, $\sum_i Q_i = \sum_j R_j$, and to constrain node outflow by assignment constraints.

Supplementary constraints can be added to (11). For instance, in an intersection with priority movements conflicting with non priority movements, the non priority flows are constrained by the priority flows. An example of such a constraint is given in Lebacque 1984 (the SSMT model), based on the gap acceptance model of Mahmassani and Sheffi 1981. Other possible constraints are constraints related to the effect of traffic lights, reducing the through-flow.

Invariance principle.

In a node model, the through flows $(Q_i)_{i\in I}$, $(R_j)_{j\in J}$ are a function of the node boundary data $(\delta_i)_{i\in I}$, $(\sigma_j)_{j\in J}$. If $Q_i(t) < \delta_i(t)$, then the link exit enters a supply regime and at t+, $\delta_i(t)$ becomes the maximum flow on link [i], i.e. $\delta_i(t+) = Q_{i,max}$. Symmetrically, if $R_j(t) < \sigma_j(t)$, then the link entry enters a demand regime and at t+, $\sigma_j(t)$ becomes the maximum flow on link [j], i.e. $\sigma_j(t+) = R_{j,max}$.

Thus the node through flows $(Q_i)_{i\in I}$, $(R_j)_{j\in J}$, as functions of the node boundary data $(\delta_i)_{i\in I}$, $(\sigma_j)_{j\in J}$, must be invariant by the following transformation:

$$\text{(12)} \qquad \left[\begin{array}{lll} \delta_i \to Q_{i,max} & \text{if} & Q_i < \delta_i \\ \sigma_j \to R_{j,max} & \text{if} & R_j < \sigma_j \end{array}\right.$$

This property will be called *invariance principle* in the sequel.
It must be satisfied, lest the K_{i0} - K_i resp. $K_j - K_{j0}$ discontinuities propagate into the wrong direction (towards the node). Such models cannot converge as the discretization step becomes vanishingly small. Node models that do not satisfy the invariance principle can only be used as discretized, phenomenological models.

The intersection model of STRADA (Buisson et al. 1995-1996) and the distribution scheme of Jin and Zhang 2002 do not satisfy the invariance principle. Let us illustrate this fact on the latter, which is simpler. The idea of the proof is the same for the STRADA model.

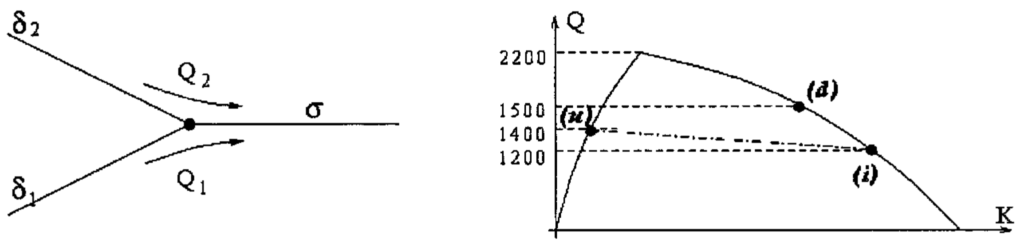


Figure 6: A merge model that does not satisfy the invariance principle

Let us consider the merge depicted in Figure 6. The flows into the merge are given by the following distribution scheme:

$$\text{(12)} \qquad Q_i = \delta_i \quad \text{if} \quad \sum_i \delta_i \le \sigma \quad \text{and} \quad Q_i = \frac{\delta_i}{\sum_k \delta_k} \quad \text{if} \quad \sum_i \delta_i > \sigma$$

Let us assume the following values, at a given time *t*:

- $\sigma = 3000$ vh / h (2 lanes), $\delta_1 = 2100$ vh / h (1 lane), $\delta_2 = 1400$ vh / h (1 lane)

These values correspond to traffic state ***(u)***. Traffic states for link 2 are depicted on the right half of Figure 6, relative to the fundamental diagram of that link.

Further we assume maximum flow values of $Q_{k,max} = 2200$ vh / h on both links [1] and [2].

Following (12), the through flows Q_i are equal to

$$Q_1 = \frac{2100}{2100+1400} 3000 = 1800 \text{ vh/h} \quad \text{and} \quad Q_2 = \frac{1400}{2100+1400} 3000 = 1200 \text{ vh/h}$$

These values correspond to traffic state (*i*). This implies that at $t+$, the supply at the exit of both links is less than demand, hence traffic states (*i*) are congested and the demand at the node becomes equal to $\delta_k = Q_{k,max}$. Both through flows now change:

$$Q_1 = \frac{2200}{2200+2200}3000 = 1500 \text{ vh/h} \quad \text{and} \quad Q_2 = \frac{2200}{2200+2200}3000 = 1500 \text{ vh/h}$$

which is traffic states (*d*) (also congested). Let us illustrate the traffic states for link [2]:

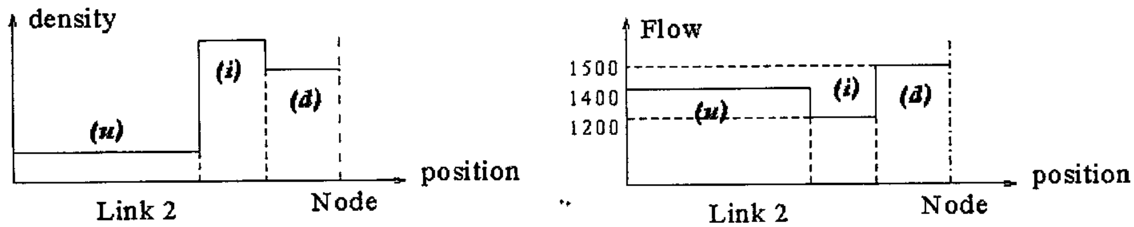


Figure 7 : Traffic states on link [2], at time $t+$

The velocity of the (*u*) → (*i*) shockwave is less in absolute value than the velocity of the (*d*) → (*i*) rarefaction wave. Thus the state (*i*) must vanish. As the velocity of the (*u*) → (*d*) shockwave is positive, the state (*d*) must vanish as well at $t+$. Thus the solution (12) must vanish at $t+$ and cannot apply.

The problem is general: the data (upstream demands and downstream supplies) are liable to be instantaneously modified by the through flows, and the modifications must be compatible with the through flows. A sufficient condition for the invariance principle to be satisfied will be given later.

EQUILIBRIUM VERSUS OPTIMIZATION MODELS FOR INTERSECTIONS.

Dynamic node model.

A simple dynamic node model, introduced in Lebacque 2003 and Lebacque and Haj-Salem 2004, and which relates to the STRADA *exchange zone* concept Buisson et al. 1995-1996, can be summarized as follows.

The node contains N vehicles and exhibits both supply $\Sigma(N)$ and demand $\Delta(N)$. These supply and demand functions are assumed to be similar to link supply and demand functions (refer to Figure 1) and express the global behavior of traffic in the node. They imply a storage capacity N_{max}, and a maximum through flow Q_{max} reached for some critical value N_{crit} of N.

A partial supply model, supported by empirical data Lebacque and Khoshyaran 2002, is the following linear split model:

(13) $$\Sigma_i(N) = \beta_i \, \Sigma(N) \quad \forall i$$

The coefficients β_i represent the share of available space in the node that is accessible to users from upstream link [i]. Typically, they should be proportional to the number of available lanes. It is possible that $\beta \overset{def}{=} \sum_i \beta_i > 1$, if there are more incoming than out coming lanes.

The partial demand for downstream link [j] can be assumed proportional to the number NO_j of vehicles exiting the node through link [j] :

(14) $$\Delta_j(N) = \frac{NO_j}{N} \Delta(N)$$

If N_{ij} denotes the number of vehicles in the node, entered through [i], bound to exit the node through [j] , the following system results, expressing conservation of vehicles [i] → [j] in the node and the usual **Min** formula for flows:

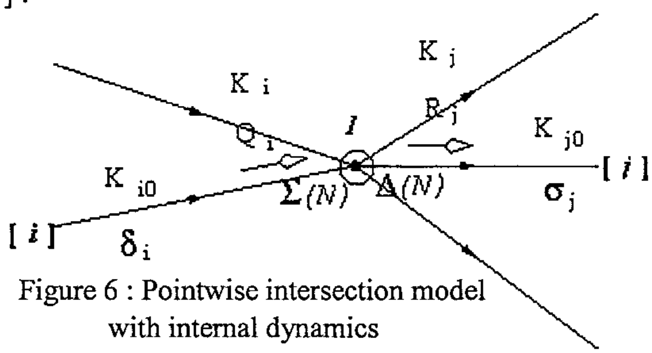


Figure 6 : Pointwise intersection model with internal dynamics

(15) $$\dot{N}_{ij} = \gamma_{ij} Q_i - \frac{N_{ij}}{NO_j} R_j \,, \quad Q_i = \mathrm{Min}\left[\delta_i, \beta_i \Sigma(N)\right], \quad R_j = \mathrm{Min}\left[\frac{NO_j}{N}\Delta(N), \sigma_j\right]$$

with $$N = \sum_{ij} N_{ij} \quad \text{and} \quad NO_j = \sum_i N_{ij}$$

The coefficients γ_{ij} denote the node turning movement coefficients (the proportion of users of link [i] which use link [j]).

The reader is referred to Lebacque and Haj-Salem 2004 for applications of this node concept to traffic hysteresis, capacity drop, speed control, and ramp metering.

Equilibrium models.

If the model scale is assumed such that N can be neglected with respect to other vehicle numbers, the dynamics can be neglected and (15) simplifies into the following equilibrium model:

(16) $$\left| \begin{array}{l} Q_i = \mathrm{Min}\left[\delta_i, \beta_i \Sigma(N)\right] \quad \forall i, \quad R_j = \mathrm{Min}\left[(NO_j / N)\Delta(N), \sigma_j\right] \quad \forall j \\ R_j - \sum_i \gamma_{ij} Q_i = 0 \quad \forall j \end{array} \right.$$

in which the N and NO_j are unknowns yielding the in and out-flows Q_i and R_j. This model can be solved because the Q_i are decreasing functions of N, the R_j are increasing functions of N, and the ratios NO_j / N essentially depend on the γ_{ij}. In this model, the quantities N and NO_j must be considered "abstract", as they are not explicit in the conservation of vehicles through the intersection, $R_j = \sum_i \gamma_{ij} Q_i \quad \forall j$

Optimization models.

An optimization model, generalizing Holden and Risebro 1995, Coclite and Piccoli 2002 and Lebacque and Khoshyaran 2002, is given by:

$$Max \sum_i \Phi_i(Q_i) + \sum_j \Psi_j(R_j)$$

(17)
$$\left| \begin{array}{l} Q_i \le \delta_i \quad \forall i, \quad R_j \le \sigma_j \quad \forall j \\ \sum_i \gamma_{ij} Q_i - R_j = 0 \quad \forall j \end{array} \right.$$

The Coclite-Piccoli model (Coclite and Piccoli 2002) for instance assumes that the total through flow is maximized (Φ_i: identity, Ψ_j: null). Here the functions Φ_i and Ψ_j are assumed increasing and strictly concave, thus the solutions of (17) are unique.
The Karush-Kuhn-Tucker optimality conditions for (17) yield:

$$\begin{array}{llll} \Phi_i'(Q_i) = \chi_i - \sum_j s_j \gamma_{ij}, & \chi_i \ge 0, \quad Q_i \le \delta_i, & \chi_i(\delta_i - Q_i) = 0 & \forall i \\ \Psi_j'(R_j) = s_j + \rho_j & \rho_j \ge 0, \quad R_j \le \sigma_j, & \rho_j(\sigma_j - R_j) = 0 & \forall j \\ \sum_i \gamma_{ij} Q_i - R_j = 0 & \forall j & & \end{array}$$

with χ_i, ρ_j, s_j the Karush-Kuhn-Tucker coefficients. The coefficients χ_i, ρ_j, can be eliminated, because of the concavity property of the the functions Φ_i and Ψ_j. Let us consider for instance Q_i. Since $\Phi_i'^{-1}$ is decreasing and $\chi_i \ge 0$ it follows:

$$Q_i = \Phi_i'^{-1}\left(-\sum_j s_j \gamma_{ij}\right) \qquad \text{if} \quad Q_i < \delta_i \text{ i.e. } \chi_i = 0,$$

$$Q_i = \delta_i = \Phi_i'^{-1}\left(-\sum_j s_j \gamma_{ij} + \chi_i\right) \le \Phi_i'^{-1}\left(-\sum_j s_j \gamma_{ij}\right) \quad \text{otherwise}$$

It results: $$Q_i = \text{Min}\left[\delta_i, \Phi_i'^{-1}\left(-\sum_l \gamma_{il} s_l\right)\right].$$

The Karush-Kuhn-Tucker optimality conditions for (17) are

(18)
$$Q_i = \mathrm{Min}\left[\delta_i, \Phi_i'^{-1}\left(-\sum_l \gamma_{il} s_l\right)\right], \quad R_j = \mathrm{Min}\left[\Psi_j'^{-1}(s_j), \sigma_j\right]$$
$$\sum_i \gamma_{ij} Q_i - R_j = 0 \quad \forall j$$

They are necessary and sufficient. The model unknowns are the s_j, i.e. the Karush-Kuhn-Tucker coefficients of the flow conservation constraints

$$\sum_i \gamma_{ij} Q_i - R_j = 0 \quad \forall j$$

The s_j can be determined by a primal Murchland-like algorithm or by solving the dual of (17).

The similarity between (16) and (18) is obvious and reflects the fact that both models satisfy the *invariance principle.* It follows from this principle that upstream demands δ_i and downstream supplies σ_j cannot be linked directly in order to yield the through flows Q_i and R_j, as for instance in Buisson et al. 1995-1996 or Jin and Zhang 2002. They can only be used in conjunction with some node supplies ϕ_i and demands ψ_j, yielding the node in- and out- flows:

(19) $$Q_i = \mathrm{Min}\left[\delta_i, \varphi_i\right], \quad R_j = \mathrm{Min}\left[\psi_j, \sigma_j\right]$$

Actually, (16) and (18) provide models of node supplies ϕ_i and demands ψ_j which are implicit functions of the upstream demands δ_i and downstream supplies σ_j . They clearly satisfy the invariance principle.

Relationship between equilibrium and optimization models.

Case of a merge. In the case of a merge: there is only one downstream link and $s_j \overset{def}{=} s$. (16) and (18) are easily shown to be rigorously equivalent in this case, by adopting the following conventions (with Δ and Σ the node demand and supply functions):

(20) $$s = N, \quad \Phi_i(Q_i) = -\int_0^{Q_i} \Sigma^{-1}(\eta / \beta_i) d\eta, \quad \Psi(R) = \int_0^R \Delta^{-1}(\eta) d\eta$$

In this case the single unknown s is determined by a simple Newton algorithm applied to (18). This result provides an interpretation of the criterion of (18).

Case of a diverge. The case of diverges is straightforward in the case of an optimization model. Indeed the constraints applied to the through flows are given by:

$Q \le \delta$, $R_j = \gamma_j Q \le \sigma_j$, hence

(21) $$Q = \mathrm{Min}\left[\delta, \mathrm{Min}_j (\sigma_j / \gamma_j)\right], \quad R_j = \gamma_j Q$$

The equilibrium model for diverges is not solved as straightforwardly. Nevertheless it can be shown that it yields the solution (21), as shown in Lebacque and Khoshyaran 2002.
The values of the ratios NO_j / N are given by $NO_j / N = \gamma_j$ if $\delta \le \mathrm{Min}_j(\sigma_j / \gamma_j)$. Under the same condition, $N = \Delta^{-1}(\delta)$.

If $\delta \ge \mathrm{Min}_j(\sigma_j / \gamma_j)$ it is necessary to define $j_0 = \mathrm{Arg}_j \mathrm{Min}_j(\sigma_j / \gamma_j)$ and N is given by $N = \Sigma^{-1}(\sigma_{j_0} / \gamma_{j_0})$. The ratios NO_j / N are given by

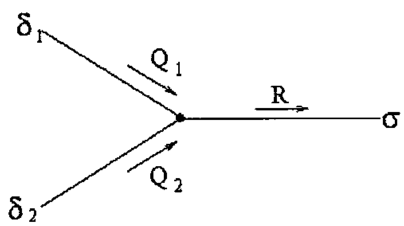
$$\frac{NO_j}{N} = \frac{\gamma_j}{\gamma_{j_0}} \frac{\sigma_{j_0}}{Q_{max}} \quad \text{if} \quad j \ne j_0 \quad \text{and} \quad \frac{NO_{j_0}}{N} = 1 - \sum_{j \ne j_0} \frac{NO_j}{N} = 1 - \frac{1-\gamma_{j_0}}{\gamma_{j_0}} \frac{\sigma_{j_0}}{Q_{max}}$$

Remark

The present approach recaptures models such as Holden and Risebro 1995, Daganzo 1995, Lebacque and Khoshyaran 2002, Coclite Piccoli 2002, Klar and Herty 2004, Lebacque and Haj-Salem 2004. Let us consider for instance the merge model in Daganzo 1995. It can be shown (based on the analysis of the following section) that the latter model is equivalent to an optimization model (17) with $\Psi = 0$ and $\Phi_i = N_{max}\left[Q_i - Q_i^2 / (2 p_i Q_{max})\right]$

STUDY OF MERGES

The object of this section is to solve completely the equilibrium model for a simple merge. It will be shown the model predicts a scatter of intersection through-flows. Let us consider the simple merge depicted in Figure 7.



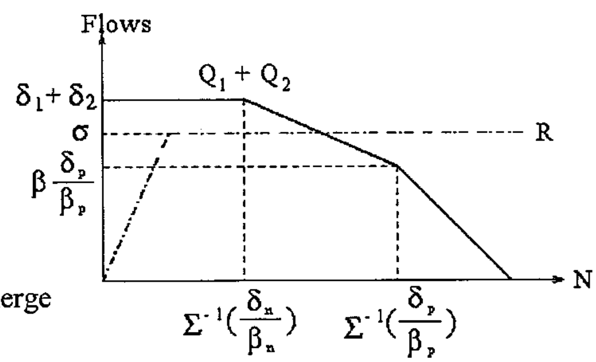


Figure 7: Study of a simple merge

The merge being described by (16), the through flows are given by:

$$\left| \begin{array}{ll} Q_i = \mathrm{Min}\left[\delta_i, \beta_i \Sigma(N)\right] & \forall i = 1,2 \\ R = \mathrm{Min}\left[\Delta(N), \sigma_j\right] & \end{array} \right. \qquad \text{and: } R - \sum_i Q_i = 0$$

The demands are ordered in the following way: $\dfrac{\delta_n}{\beta_n} \ge \dfrac{\delta_p}{\beta_p}$

The total inflow, $Q_1 + Q_2$ is a decreasing function of N.

$$Q_1 + Q_2 = \begin{cases} \sum_i \delta_i & \text{if } N \le \Sigma^{-1}(\delta_n / \beta_n) \\ \delta_p + \beta_n \Sigma(N) & \text{if } \Sigma^{-1}(\delta_n / \beta_n) \le N \le \Sigma^{-1}(\delta_p / \beta_p) \\ \beta \Sigma(N) & \text{if } \Sigma^{-1}(\delta_p / \beta_p) \le N \end{cases}$$

R is an increasing function of N. Thus the conservation equation $R = Q_1 + Q_2$ admits a solution in N which can be expressed analytically by considering the different possible expressions of $Q_1 + Q_2$. The right side of Figure 7 illustrates this process.

The values of N are:

- If $\sigma \ge \delta_1 + \delta_2$, $N = \Delta^{-1}(\delta_1 + \delta_2)$
- If $\beta(\delta_p / \beta_p) \le \sigma \le \delta_1 + \delta_2$, then $N = \mathrm{Arg}_v \lfloor \delta_p + \beta_n \Sigma(v) = \sigma \rfloor$, i.e. $N = \Sigma^{-1}((\sigma - \delta_p) / \beta_n)$
- If $\sigma \le \beta(\delta_p / \beta_p)$, then $N = \Sigma^{-1}(\sigma / \beta)$

The following values of the through-flows result

- If $\sigma \ge \delta_1 + \delta_2$, then $Q_i = \delta_i$, $R = \delta_1 + \delta_2 = Q_1 + Q_2$
- If $\beta(\delta_p / \beta_p) \le \sigma \le \delta_1 + \delta_2$, then $Q_p = \delta_p$, $Q_n = \sigma - \delta_p$, $R = \sigma = Q_1 + Q_2$
- If $\sigma \le \beta(\delta_p / \beta_p)$, then $Q_i = \sigma(\beta_i / \beta)$, $R = \sigma = Q_1 + Q_2$

The through-flows depend on the upstream demands and downstream supply, therefore they cannot be related by a functional relationship, except in the oversaturated case (when $\sigma \le \beta(\delta_p / \beta_p)$). Therefore the model predicts a wide scatter of the flow plot.

It can be shown that by eliminating the supply and demand from the above expressions, *linear constraints* result for the through-flows. It follows that the geometrical properties of the scatter plot will not be affected by the sampling interval.

Measurements (A4 motorway in the vicinity of Paris, see figure above) and computer simulation data (see figure 8 below) generated according to the equilibrium merge model illustrate this observation. Experimental data and model predictions are compatible.
Oltra and Jardin 1996 reported similar experimental scatter of through-flow data.

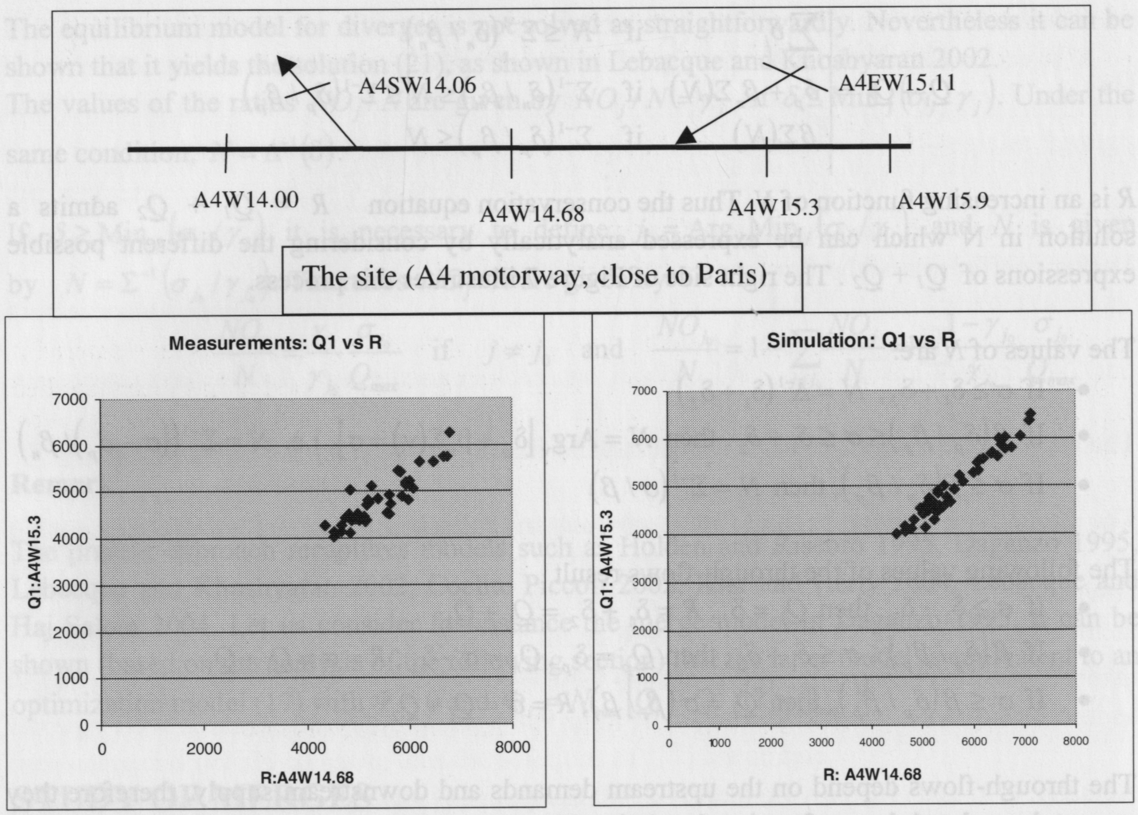


Figure 8: measurements vs. predictions of the equilibrium merge model

MULTICOMMODITY FIFO LWR MODEL

Introduction

In order to complete the description of traffic flow on the network, we introduce the FIFO multicommodity LWR model. In this model, users are disaggregated according to some criterion d, representing the destination, or origin-destination, or path, or user class... The criterion d will be noted as a superscript Thus both density and flow are split according to d.

$$K(x,t) = \sum_{d=1\ldots D} K^d(x,t) \quad \forall x,t \quad \text{and} \quad Q(x,t) = \sum_{d=1\ldots D} Q^d(x,t) \quad \forall x,t$$

By the conservation of vehicles,

(22) $$\frac{\partial K^d}{\partial t} + \frac{\partial Q^d}{\partial x} = 0 \quad \forall d = 1\ldots D$$

The specificity of the FIFO multicommodity LWR model is that the velocity of vehicles is assumed to be independent of d. Thus all vehicles have the same velocity

$$V^d(x,t) \overset{def}{=} V(x,t) = V_e(K(x,t),x) \quad \forall x,t, \quad \text{and} \quad \forall d = 1 \dots D$$

Let us introduce the *composition coefficients* χ^d:

$$Q^d = K^d V \quad \text{and} \quad K^d = \chi^d K, Q^d = \chi^d Q \quad \forall d$$

Traffic flow is homogeneous, and vehicles leave any link in the same order as they entered it, regardless of their attribute d (hence the name "FIFO"). This model has been studied, especially in its numerical aspects, by several authors, among which Buisson *et al.* 1995-1996, Jin and Zhang 2004, Lebacque and Khoshyaran 2001 for instance.

The equations of the multicommodity traffic flow are (under conservation form) given by:

(23) $$\frac{\partial \mathbf{K}}{\partial t} + \frac{\partial}{\partial x} F(\mathbf{K}) = 0$$

- $\mathbf{K} \overset{def}{=} (K^d)_{d=1 \dots D}$ and $F(K) = V_e(K)\mathbf{K}$
- $K \overset{def}{=} \sum_{d=1 \dots D} K^d = E^+ . \mathbf{K}$ where $E^+ = (1 \dots 1)$ and $^+$ means transpose

Analysis: Rarefaction waves, contact discontinuities, shockwaves, Riemann problem

After some straightforward algebra, it follows from (22) that the composition coefficients χ^d satisfy the following advection equation and thus are constant along vehicle trajectories, a fact noted by several authors.

(24) $$\frac{\partial \chi^d}{\partial t} + V_e(K)\frac{\partial \chi^d}{\partial x} = 0 \quad \forall d = 1 \dots D$$

The gradient of F, $A(\mathbf{K}) \overset{def}{=} \nabla_{\mathbf{K}} F(\mathbf{K}) = V_e(K)\mathbf{Id} + V_e{}'(K)\mathbf{K}.E^+$ with **Id** referring to the identity matrix, admits two eigenvalues, $\lambda_+ \geq \lambda_-$.

- $\lambda_- = Q_e{}'(K)$, corresponding to a one-dimensional eigenspace $X_- = (\mathbf{K})$
- $\lambda_+ = V_e(K)$, corresponding to a D-1 - dimensional eigenspace $X_+ = \{h / E^+.h = 0\}$.

Rarefaction waves. The λ_- - field is completely non-linear, since $\nabla_{\mathbf{K}} \lambda_- = Q_e{}''(K)E$. We choose as representative of X_- a vector R_- along which λ_- increases:

$$R_- = -\mathbf{K} \quad \text{thus} \quad \nabla_{\mathbf{K}} \lambda_- . R_- = -Q_e{}''(K)K > 0$$

Integral curves of R_- are trivial (straight lines to the origin). If we consider a traffic state $\mathbf{K_0}$, the traffic states $\mathbf{K}$ that can be connected to $\mathbf{K_0}$ by a rarefaction wave are such that $\mathbf{K} = \varsigma\mathbf{K}_0$ with ς a real number such that : $0 < \varsigma < 1$.

Contact discontinuities. The eigenvalue λ_- is linearly degenerate. Indeed, $\nabla_\mathbf{K}\lambda_+ = V_e'(K)E$, thus $\nabla_\mathbf{K}\lambda_+ .h = V_e'(K)E^+.h = 0 \quad \forall h \in X_+$.

Let us we consider a traffic state $\mathbf{K_0}$. The traffic states $\mathbf{K}$ that can be connected to $\mathbf{K_0}$ by a contact discontinuity lie in the integral manifold of X_+ which contains $\mathbf{K_0}$, that is $\mathbf{K_0} + X_+$. It follows that $K = E^+.\mathbf{K} = E^+.\mathbf{K}_0 = K_0$. The traffic states $\mathbf{K}$ and $\mathbf{K_0}$ have the same total density $K=K_0$ and differ only by their composition.

Shockwaves. The traffic states $\mathbf{K}$ that can be connected to a given traffic state $\mathbf{K_0}$ by a shockwave are such that for some real number s, depending on $\mathbf{K}$ and $\mathbf{K_0}$:

$$F(\mathbf{K}) - F(\mathbf{K}_0) = s(\mathbf{K} - \mathbf{K}_0) \tag{25}$$

(Rankine-Hugoniot). It follows that s and $\mathbf{K} - \mathbf{K_0}$ satisfy $B(\mathbf{K}-\mathbf{K}_0) = s(\mathbf{K}-\mathbf{K}_0)$, i.e. are eigenvalue and eigenvector of the operator B (depending on $\mathbf{K}$ and $\mathbf{K_0}$) defined as:

$$B = \int_0^1 A(\mathbf{K}_0 + t(\mathbf{K}-\mathbf{K}_0))dt$$

By the Lax entropy condition, when $\mathbf{K}$ tends towards $\mathbf{K_0}$ and B tends towards A, s should tend towards $\lambda_-(\mathbf{K_0})$ and $\mathbf{K} - \mathbf{K_0}$ should tend towards $R_-(\mathbf{K_0})$ in direction and be opposed to $R_-(\mathbf{K_0})$. Some tedious but easy calculations yield the following expression for B:

$$B = \alpha\mathbf{Id} + b.E^+ \quad \text{with} \quad \left| \begin{array}{l} \alpha = \dfrac{W_e(K) - W_e(K_0)}{K - K_0} \\ b = \dfrac{V_e(K)\mathbf{K} - V_e(K_0)\mathbf{K}_0}{K - K_0} - \dfrac{W_e(K) - W_e(K_0)}{(K-K_0)^2}(\mathbf{K} - \mathbf{K}_0) \end{array} \right. \tag{26}$$

and W_e any primitive of V_e. The structure of B is the same as the structure of A, and the eigen-elements of B result immediately. B admits two eigenvalues,

$$s_- = \frac{Q_e(K) - Q_e(K_0)}{K - K_0} \quad \text{and} \quad s_+ = \frac{W_e(K) - W_e(K_0)}{K - K_0}$$

corresponding respectively to the one-dimensional eigenspace $Y_- = (b)$ and to the

D-1 - dimensional eigenspace: $Y_+ = X_+ = \{h / E^+.h = 0\}$.

If the traffic state $\mathbf{K}$ can be connected to the traffic state $\mathbf{K_0}$ by a shockwave, $\mathbf{K} - \mathbf{K_0}$ is collinear to b, thus $\mathbf{K}$ is collinear to $\mathbf{K_0}$. In view of the Lax entropy condition, if $\mathbf{K}$ is connected to $\mathbf{K_0}$ by a λ_--shockwave, then $\mathbf{K} = \varsigma\mathbf{K}_0 \quad with \quad \varsigma > 1$.

The shockwaves and rarefaction waves of the FIFO multicommodity LWR model are the ordinary waves of the underlying LWR model.

Inhomogeneous Riemann problem for the FIFO LWR model

Let us consider an inhomogeneous Riemann problem, defined by the traffic states on the left $\mathbf{K}_\ell$ and on the right $\mathbf{K}_r$, and by the fundamental diagrams $Q_e(.,\ell)$ and $Q_e(.,r)$ on the left and on the right. Since there are only two eigenspaces which are transversal, and since contact discontinuities propagate at >0 speed, the solution of the Riemann problem will yield one intermediate state $\mathbf{K}_0$ which has the same composition as $\mathbf{K}_\ell$ and the same total density as $\mathbf{K}_r$:

$$K_0 = E^+.\mathbf{K}_0 = E^+.\mathbf{K}_r = K_r\,, \quad (1/K_\ell)\mathbf{K}_\ell = (1/K_r)\mathbf{K}_0\,.$$

$\mathbf{K}_0$ is connected to $\mathbf{K}_r$ by a contact singularity with velocity $V_r = V_e(K_r, r)$.

By multiplying equation (23) on the left by E+, (1) results. The inhomogeneous Riemann problem is solved for the total density $K = E^+.\mathbf{K}$ in the usual way (as in Lebacque 1996).

In the *(x,t)* plane, the traffic composition is equal to the composition of $\mathbf{K}_\ell$ for all points such that $x < V_r t$ and equal to the composition of $\mathbf{K}_r$ for all points such that $x > V_r t$.

Network modeling in the FIFO multicommodity LWR model

Flow on links and boundary conditions

Flow on links. The total density K follows the LWR model, equation (1). The compositions follow the advection equation (22). The partial flows follow (23).
The contact singularity associated to a composition discontinuity always *propagates at the speed of the right-hand-side conditions.* Thus at a location x at time t , a composition discontinuity propagates at velocity $V_e(K(x+,t), x+)$ (with V_e the equilibrium speed). Discontinuities must be treated as a Riemann problem.

Boundary conditions apply notably at entry and exit points of the network. The preceding subsection on the Riemann problem enables us to give the boundary conditions for the FIFO multicommodity LWR model.

- The total density K follows the LWR model, i.e. equation (1). Therefore the boundary conditions for K are upstream supply and downstream demand, with boundary total flows and densities given by (4) and (5).

- For composition coefficients, the boundary data are upstream composition coefficients in the case of upstream boundary points (entry points).
- There are no downstream boundary data for composition coefficients.

For an entry point (*a*), the boundary data are the upstream demand $\Delta_u(t)$ and upstream composition coefficients $\chi^d_u(t)$. Thus the total and partial inflows are given by:

$$Q(a,t) = \mathrm{Min}\left[\Delta_u(t), \Sigma(a,t)\right]$$
$$\chi^d(a,t) = \chi^d_u(t), \quad Q^d(a,t) = \chi^d(a,t)Q(a,t) \quad \forall d$$

The density is given by (5):

$$K(a,t+) = \begin{bmatrix} \Delta_e^{-1}(\Delta_u(t)) & \text{if} \quad \Delta_u(t) < \Sigma(a,t) \\ K(a,t) & \text{if} \quad \Delta_u(t) \geq \Sigma(a,t) \end{bmatrix}$$

For an exit point (*b*), the boundary data are the downstream supply $\Sigma_s(t)$. Thus the total and partial inflows are given by:

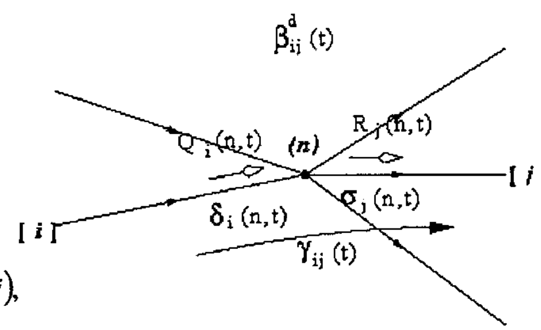
$$Q(b,t) = \mathrm{Min}\left[\Delta(b,t), \Sigma_s(t)\right], \quad Q^d(b,t) = \chi^d(b,t)Q(b,t) \quad \forall d$$

The density is given by (5):

$$K(b,t+) = \begin{bmatrix} \Sigma_e^{-1}(\Sigma_s(t)) & \text{if} \quad \Sigma_s(t) < \Delta(b,t) \\ K(b,t) & \text{if} \quad \Sigma_s(t) \geq \Delta(b,t) \end{bmatrix}$$

Intersections, fixed discontinuities.

Depending on the model chosen, the *intersection through-flows* are calculated according to (16) or (17)-(18). The node demands and supplies are calculated as in (2):
$\delta_i(t) = \Delta_e(K(n-,t;i),n;i), \quad \sigma_j(t) = \Sigma_e(K(n+,t;j),$



The node turning movement coefficients $\gamma_{ij}(t)$ are determined using two kinds of information, the composition of traffic exiting links [i] $\chi^d(n,t;i)$, and the assignment coefficients $\beta^d_{ij}(t)$ (fraction of users exiting link [i] with final destination (or class) d, intending to use link [j]). The assignment coefficients are exogenous to the model and must be calculated separately (shortest path, variable message sign, user choice model…).

$$\gamma_{ij}(t) = \sum_d \beta_{ij}^d(t)\chi^d(n,t;i)$$

The densities at the intersection node are calculated according to (5):

$$K(n,t+;i) = \begin{bmatrix} \Sigma_e^{-1}(Q(n,t;i),n;i) & \text{if} \quad Q(n,t;i) < \delta_i(t) \\ K(n,t;i) & \text{if} \quad Q(n,t;i) \geq \delta_i(t) \end{bmatrix}$$

$$K(n,t+;j) = \begin{bmatrix} \Delta_e^{-1}(R(n,t;j),n;j) & \text{if} \quad R(n,t;j) < \sigma_j(t) \\ K(n,t;j) & \text{if} \quad R(n,t;j) \geq \sigma_j(t) \end{bmatrix}$$

Consider now a *fixed discontinuity*, i.e. Q_e is discontinuous at some location. Fixed discontinuities occur at locations where the infrastructure geometry or the traffic regulations (speed limits) change. They can be treated as elementary intersections of the optimization type, in which the total through-flow is maximized (i.e. the *min formula* (2) and (3)).

The usual rules of the LWR model apply to the calculation of total densities. By the Rankine-Hugoniot formula (25), all flows (partial and total) are conserved through the discontinuity, the velocity of which is null. The traffic composition is conserved through the discontinuity.

CONCLUSION

Based on the BLN boundary conditions, the paper proves that local traffic supply and demand define the boundary conditions for the LWR model. A rigorous methodology for intersection modeling within the LWR framework is proposed, based on the supply/demand concept and the Riemann problem for nodes. The invariance principle is shown to be necessary for intersection models to be consistent. Two models, the equilibrium and the optimization model, are introduced which satisfy the invariance principle. Analytical solutions are given for the merge and diverge models, and in these special cases optimization and equilibrium model are shown to be equivalent and analytically tractable. The merge model is shown to be compatible with experimental data, and to predict wide scatter of flow plots.

Finally, the analysis of the Riemann problem for the multicommodity FIFO LWR model makes it possible to specify composition dynamics and boundary data. Thus the intersection models, the boundary data and the flow models can be combined into a single network traffic flow model. Future research should include modeling of heterogeneous traffic for non FIFO intersection modeling, as well as carrying out experimental tests for intersection models.

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APPENDIX: SKETCH OF THE PROOF OF (9), (10)

We shall check in this section, in the case of left-hand-side (upstream boundary conditions), that (6) is equivalent to an upstream demand boundary condition. More precisely the BLN boundary data $A(a,t)$ is *equivalent to a demand boundary data* $\Delta_e(A(a,t),a)$. Although some algebra is required, proving the equivalence is relatively straightforward.

Symmetrically, in the case of right-hand-side (downstream) boundary conditions, the BLN boundary data $A(b,t)$ is *equivalent to a supply boundary data* $\Sigma_e(A(b,t),b)$. The proof is analogue to the proof given below for upstream boundary conditions.

First, using the properties of properties of the *sgn* (sign) function, we rewrite (6) as follows

(A1) $$\{\mathrm{sgn}[K-\kappa]-\mathrm{sgn}[A-\kappa]\}[Q_e(K)-Q_e(\kappa)].n \geq 0 \qquad \forall \kappa \in I(a,b)$$

with $$I(A,K) \overset{def}{=} \begin{cases} [A,K] & \text{if } A \leq K \\ [K,A] & \text{if } K \leq A \end{cases}$$

Note that we skip the time dependency of variables: K means $K(a+,t)$ with a the upstream extremity of the link, and A stands for $A(t)$. n stands for $n(a)$.

The **first point** is that:

$$\{\mathrm{sgn}[K-\kappa]-\mathrm{sgn}[A-\kappa]\} = \begin{cases} 2\mathrm{sgn}[K-A] & \text{if} \quad \kappa \in I(A,K) \\ 0 & \text{if} \quad \kappa \notin I(A,K) \end{cases} \qquad \forall \kappa \in I(a,b)$$

The **second point** is that n=-1 for upstream boundary conditions, thus (A1) is equivalent to:

$$\mathrm{sgn}[K-A][Q_e(K)-Q_e(\kappa)] \leq 0 \qquad \forall \kappa \in I(a,b)$$

which translates into:

(A2) $$\begin{cases} Q_e(K) = \mathrm{Min}_{\kappa \in [A,K]} Q_e(\kappa) & \text{if} \quad K \geq A \\ Q_e(K) = \mathrm{Max}_{\kappa \in [K,A]} Q_e(\kappa) & \text{if} \quad K \leq A \end{cases}$$

A^* denotes the *conjugate* of A with respect to Q_e : A^* must satisfy both $Q_e(A) = Q_e(A^*)$ and $A \neq A^*$. Let us recall that K_{max}, K_{crit} denote respectively the maximum and critical density.

Let us analyze (A2).

1. If $K \le A \le K_{crit}$, then: $\mathrm{Max}_{\kappa\in[K,A]} Q_e(\kappa) = Q_e(A)$ and it follows from (A2) that $K=A$.

2. If $K \le K_{crit} \le A$ then: $\mathrm{Max}_{\kappa\in[K,A]} Q_e(\kappa) = Q_{max}$ and it follows from (A2) that $K=K_{crit}$.

3. If $K_{crit} \le K \le A$ then: $\mathrm{Max}_{\kappa\in[K,A]} Q_e(\kappa) = Q_e(K)$ in accordance with (A2).

4. If $K_{crit} \le A \le K$, then: $\mathrm{Min}_{\kappa\in[A,K]} Q_e(\kappa) = Q_e(K)$ is satisfied because Q_e is decreasing on $\lfloor K_{crit}, K_{max} \rfloor$.

5. If $A \le K_{crit} \le K$, then $Q_e(K) \le Q_e(A)$ implies that $K \ge A^*$.

6. If $A \le K \le K_{crit}$, then $Q_e(K) \le Q_e(A)$ implies that $K = A$.

Summarizing this discussion, (A2) is shown to be equivalent to:

(A3) $\quad A \le K_{crit}: \; K \in \{A\} \cup [A^*, K_{max}]$ and $A \ge K_{crit}: \; K \in [K_{crit}, K_{max}]$

A cannot be prescribed: K is $\neq A$ in most cases. This fact is a consequence of the way information propagates in the LWR model. The propagation velocity depends on the traffic state. Let us **interpret the cases** in which K is $\neq A$.

- $A \le K_{crit}$ and $K \in \lfloor A^*, K_{max})$: $\quad \Sigma_e(K) = Q_e(K) \le Q_e(A) = \Delta_e(A)$
- $A \ge K_{crit}$ and $K \in \lfloor K_{crit}, K_{max})$: $\quad \Sigma_e(K) = Q_e(K) \le Q_{max} = \Delta_e(A)$
-

All these cases are characterized by: $\Sigma_e(K) \le \Delta_e(A)$ (*supply regime*).

The case $A \le K_{crit}$, and $K = A$, corresponds to a *demand regime*:

$$\Delta_e(A) = Q_e(A) \le Q_{max} = \Sigma_e(K)$$

In other words, the boundary data A in the sense of BLN is equivalent to the boundary data $\Delta_e(A)$ in the supply-demand sense.

In the *case of downstream boundary conditions* (i.e. $A = A(b,t)$, $K = K(b\text{-},t)$), the following admissible values of K result:

(A4) $\quad A \le K_{crit}: \; K \in [0, K_{crit}]$ and $A \ge K_{crit}: \; K \in \{A\} \cup [0, A^*]$